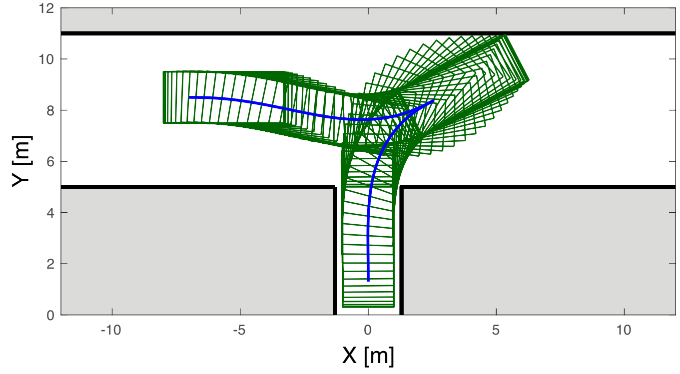
Optimization-Based Collision Avoidance

Alexander Liniger

George Zhang and Francesco Borrelli

IfA Coffee Talk



ETH zürich
Berkeley
UNIVERSITY OF CALIFORNIA
Institut für Automatik
Automatic Control Laboratory
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Motivation

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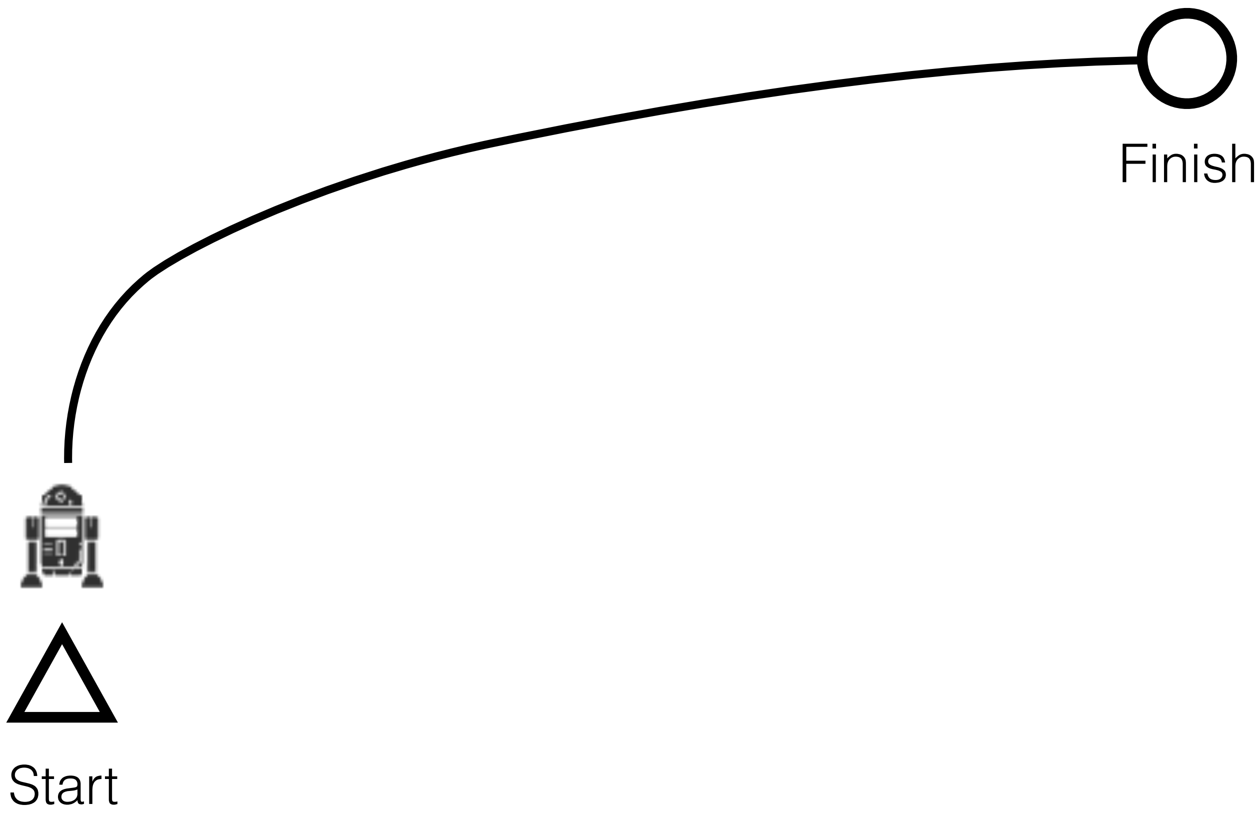

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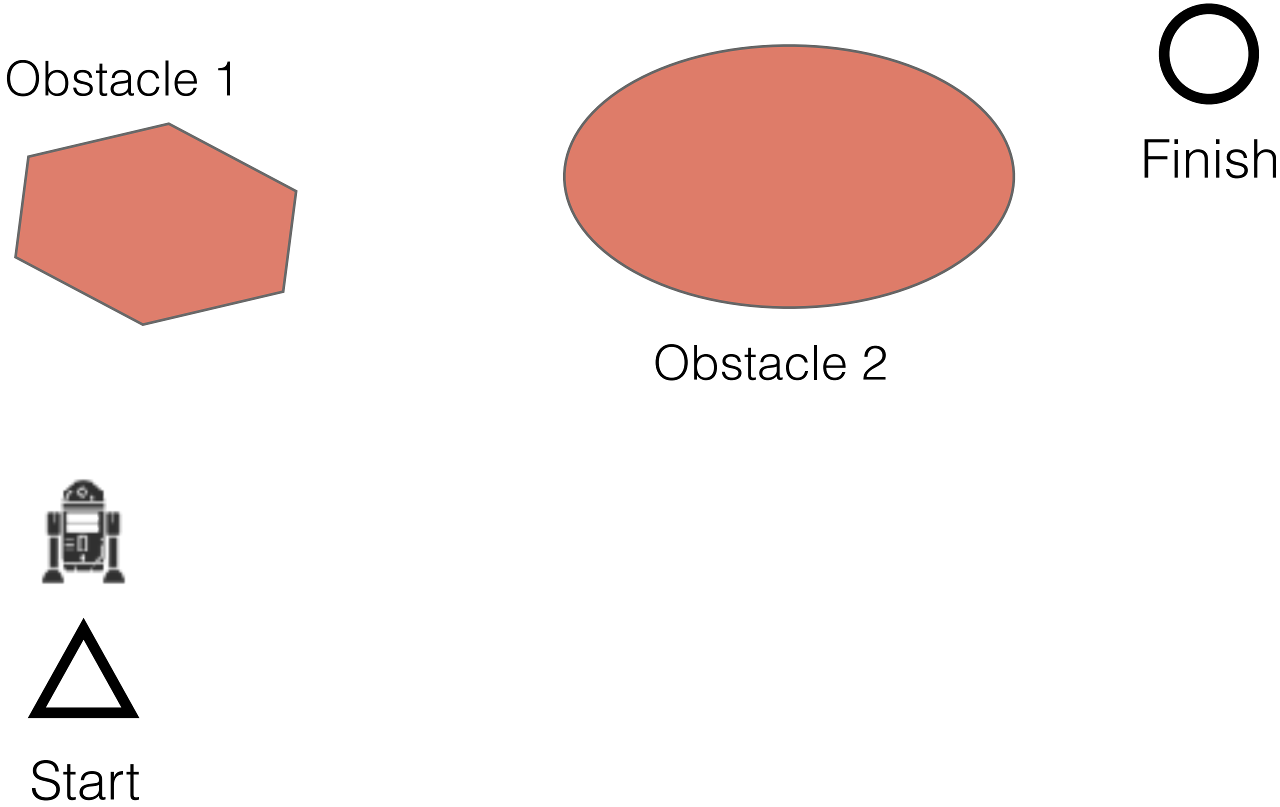

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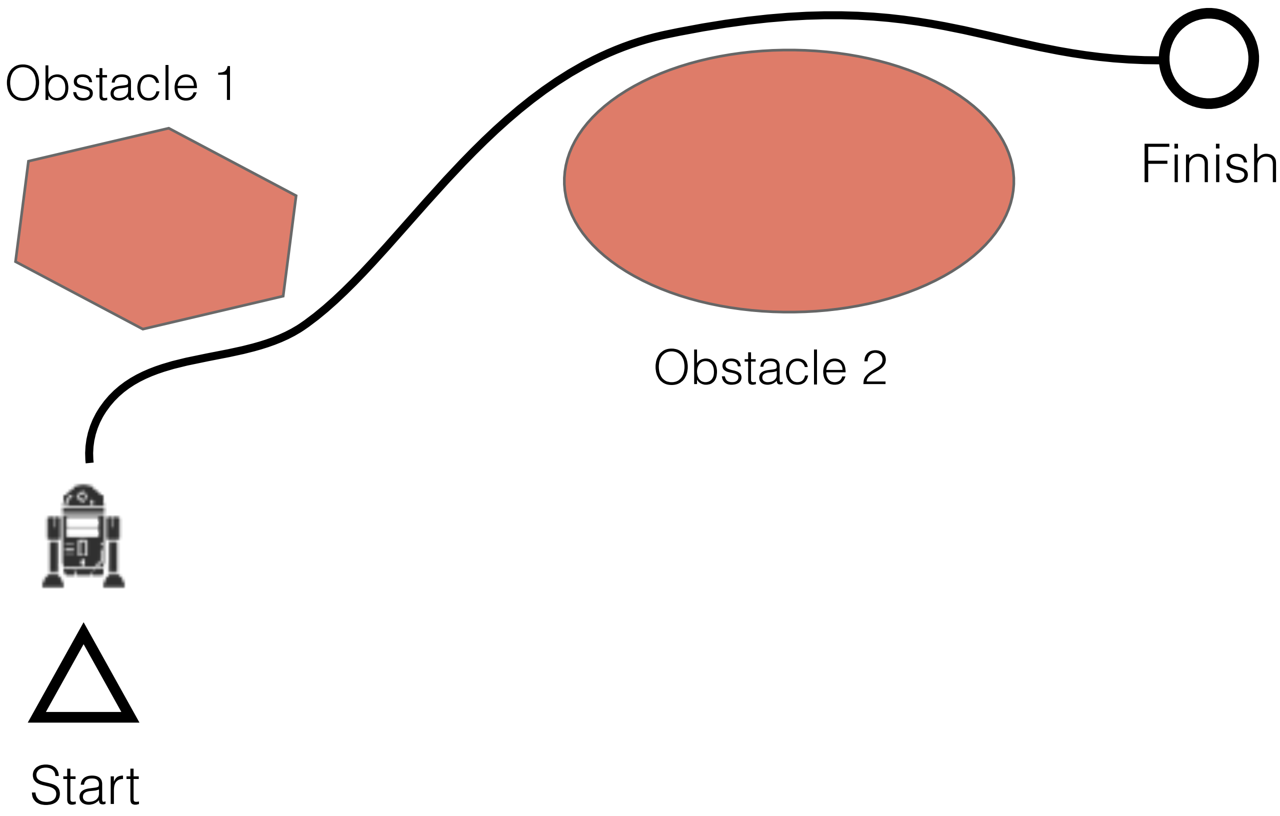

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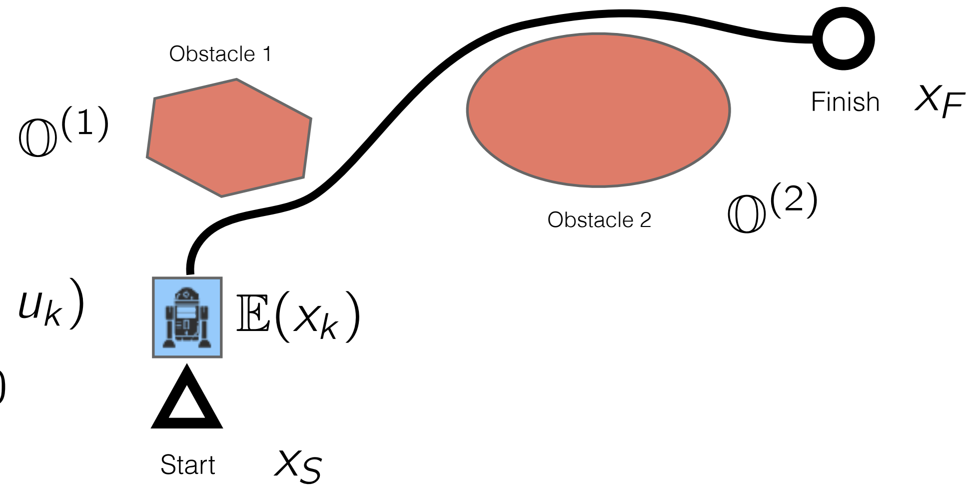
$$h(x_k, u_k) \leq 0$$



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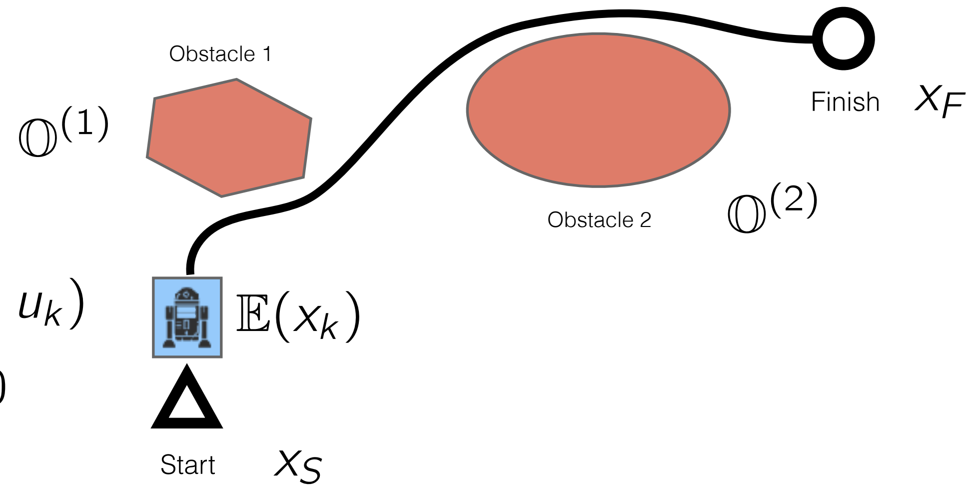

Motion planning optimization problem

$$\begin{aligned}
\min_{x,u} \quad & \sum_{k=0}^{N} \ell(x_k, u_k) && \text{Cost} \\
\text{s.t.} \quad & x_0 = x_S, \; x_{N+1} = x_F && \text{Start and finish state} \\
& \left.\begin{aligned} & x_{k+1} = f(x_k, u_k), \\ & h(x_k, u_k) \leq 0, \\ & \mathbb{E}(x_k) \cap \mathbb{O}^{(m)} = \emptyset, \end{aligned}\right\} \begin{aligned} & k = 0, \dots, N, \\ & m = 1, \dots, M, \end{aligned} && \begin{aligned} & \text{Dynamics} \\ & \text{Collision constraints} \end{aligned}
\end{aligned}$$

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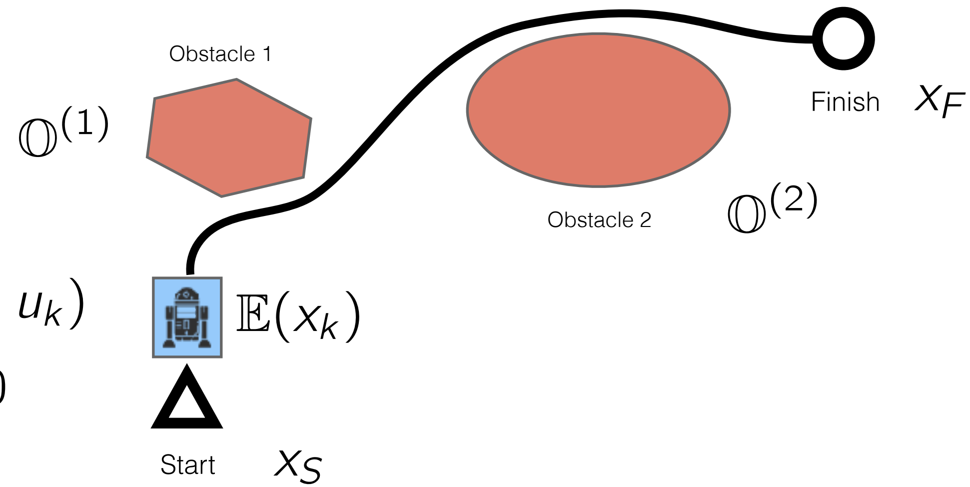

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Signed distance

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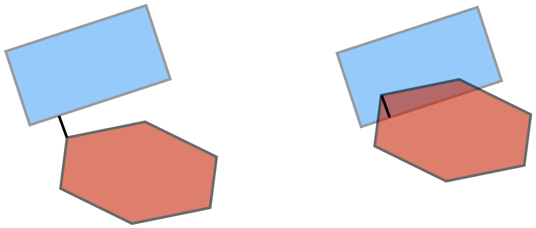
$$\mathrm{sd}(\mathbb{E}(x), \mathbb{O}) := \mathrm{dist}(\mathbb{E}(x), \mathbb{O}) - \mathrm{pen}(\mathbb{E}(x), \mathbb{O})$$

Collision constraints reformulation

$$\mathbb{E}(x) \cap \mathbb{O} = \emptyset \;\Leftrightarrow\; \mathrm{sd}(\mathbb{E}(x), \mathbb{O}) > 0$$



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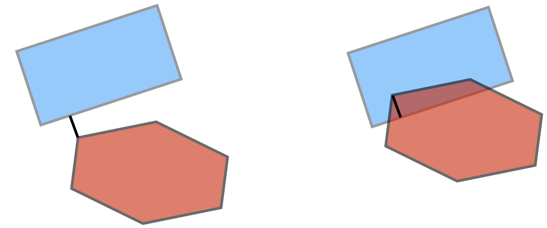

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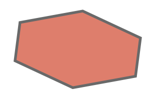
- Obstacles
  - Convex obstacles

$$\mathbb{O}^{(m)} = \{y \in \mathbb{R}^n \colon A^{(m)} y \preceq_{\mathcal{K}} b^{(m)}\}$$



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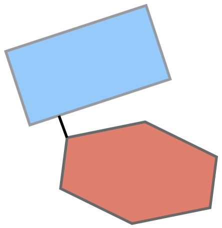

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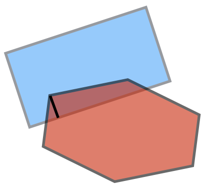

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- Obstacles
  - Convex obstacles
  - Union of convex sets can well approximate non-convex sets

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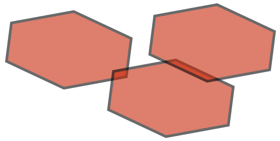

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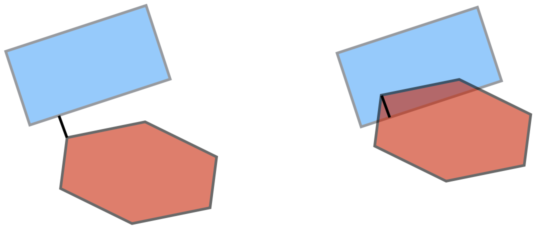

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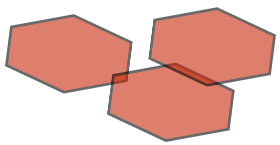

- Ego shape

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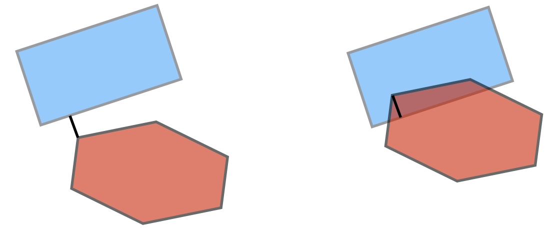

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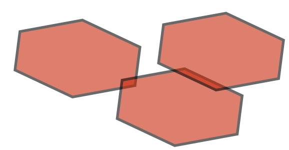

- Ego shape
  - Point mass

$$\mathbb{E}(x_k) = p(x_k)$$

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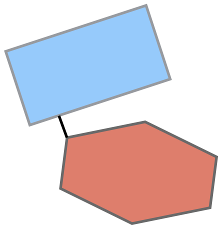

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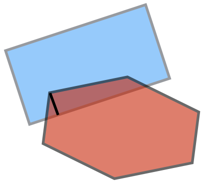

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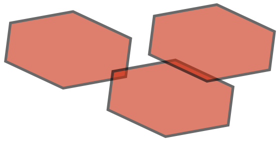
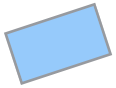


- Ego shape
  - Point mass $\quad \mathbb{E}(x_k) = p(x_k)$
  - Full-sized



$$\mathbb{E}(x_k) = R(x_k)\mathbb{B} + t(x_k), \quad \mathbb{B} := \{y \colon Gy \preceq_{\bar{\mathcal{K}}} g\}$$

Point mass collision avoidance

$$x_{k+1} = f(x_k, u_k)$$

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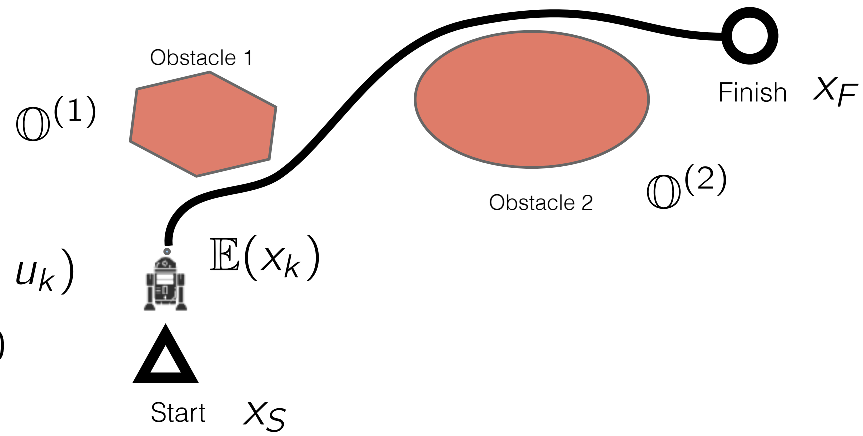

Smooth collision constraint reformulation

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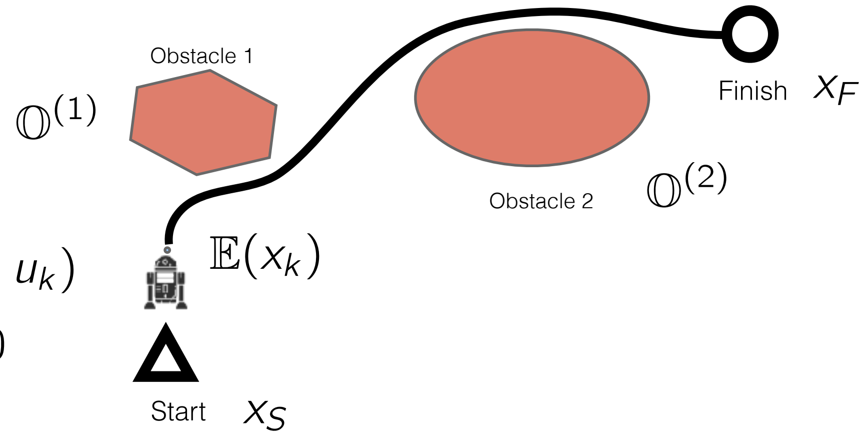
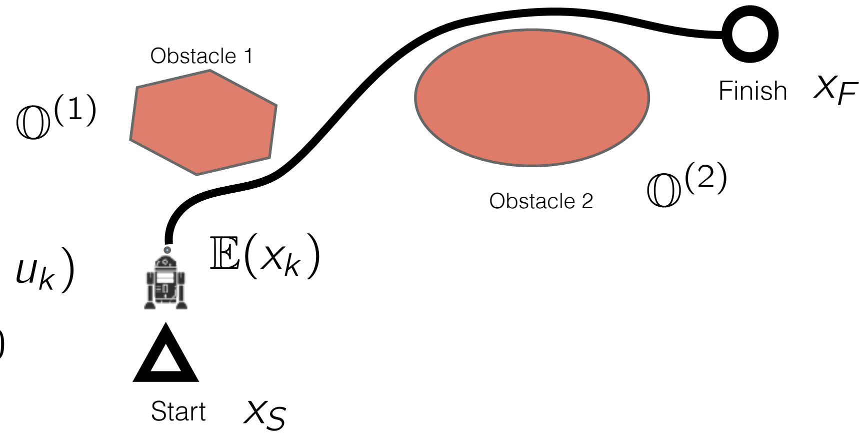


Smooth collision constraint reformulation

- Point mass ego shape $\mathbb{E}(x_k) = p(x_k)$ -> extract position from state

Point mass collision avoidance



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- When is it “easy” to handle the collision constraint $\mathbb{E}(x) \cap \mathbb{O} = \emptyset$?

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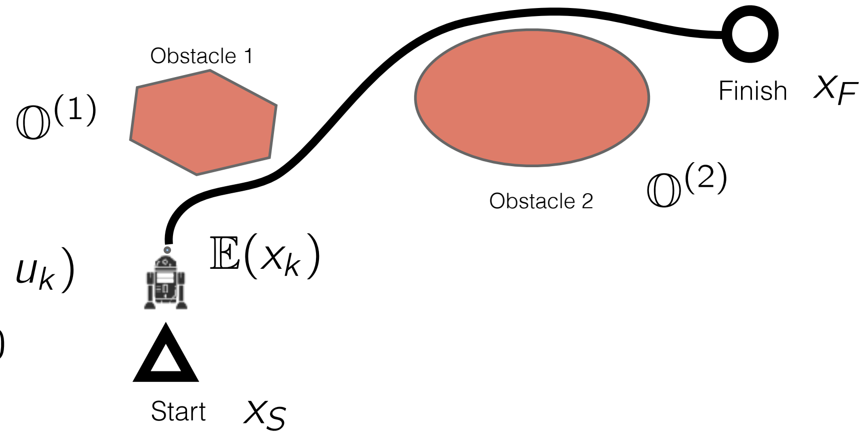

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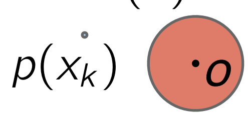

  - Polytopes + linear dynamics -> mixed integer or disjunctive programming

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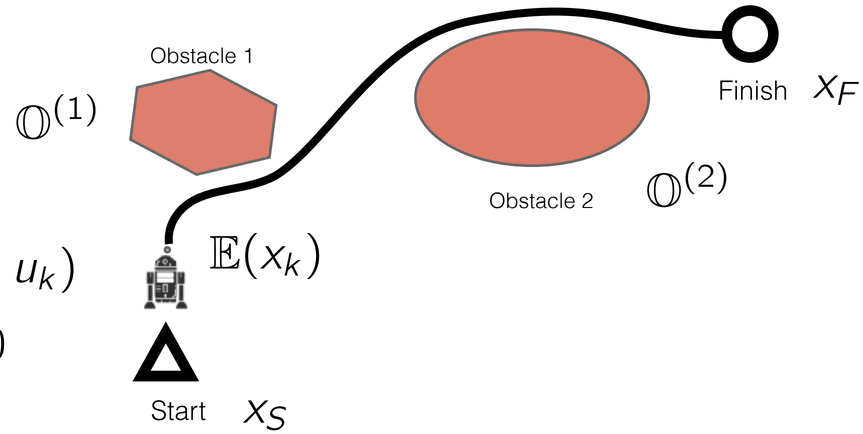

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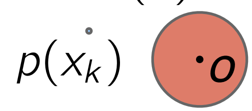

  - Polytopes + linear dynamics -> mixed integer or disjunctive programming
- We show that the collision constraint can be reformulated as a smooth but non-convex constraint by reformulating the distance and signed distance

$$\mathbb{E}(x) \cap \mathbb{O} = \emptyset \;\Leftrightarrow\; \mathrm{dist}(\mathbb{E}(x), \mathbb{O}) > 0 \qquad \mathbb{E}(x) \cap \mathbb{O} = \emptyset \;\Leftrightarrow\; \mathrm{sd}(\mathbb{E}(x), \mathbb{O}) > 0$$

Distance reformulation

Theorem 1: Distance reformulation

If the obstacle is given by $\mathbb{O} = \{y \in \mathbb{R}^n \colon Ay \preceq_{\mathcal{K}} b\}$ and the ego shape is a point mass $\mathbb{E}(x) = p$, the requirement that the distance between the two sets is larger than a safety distance $\mathsf{d}_{\mathrm{min}} \geq 0$ is equivalent to the following constraints:

$$\mathrm{dist}(\mathbb{E}(x), \mathbb{O}) > \mathsf{d}_{\mathrm{min}} \iff \exists \lambda \succeq_{\mathcal{K}^*} 0 \colon (A\,p - b)^\top \lambda > \mathsf{d}_{\mathrm{min}}, \ \|A^\top \lambda\|_* \leq 1$$

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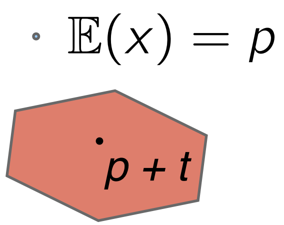

$$\operatorname{dist}(\mathbb{E}(x), \mathbb{O}) = \min_t \{\|t\| : A(\mathbb{E}(x) + t) \preceq_{\mathcal{K}} b\}$$

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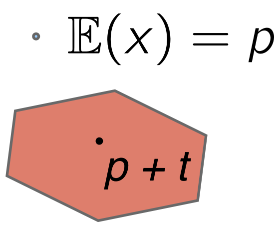

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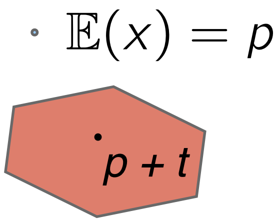
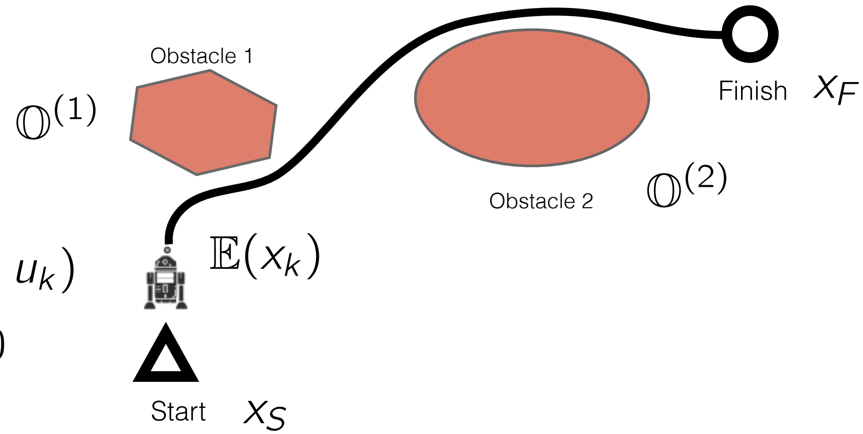
  - If there exists a λ which fulfils these conditions the distance constraint is fulfilled



Distance Reformulation



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Collision free motion planning optimization problem

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& (A^{(m)} p_k - b^{(m)})^\top \lambda_k^{(m)} > 0, && \\
& \|{A^{(m)}}^\top \lambda_k^{(m)}\|_* \leq 1,\ \lambda_k^{(m)} \succeq_{\mathcal{K}^*} 0, && \text{Collision constraints} \\
& \text{for } k = 0, \ldots, N,\ m = 1, \ldots, M, &&
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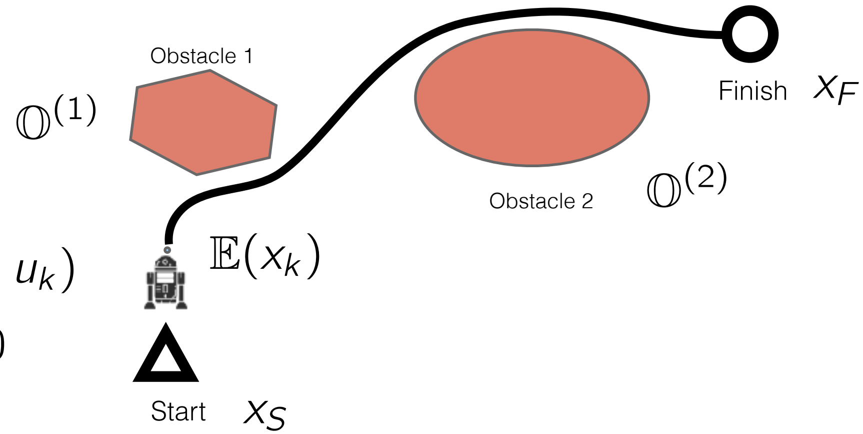

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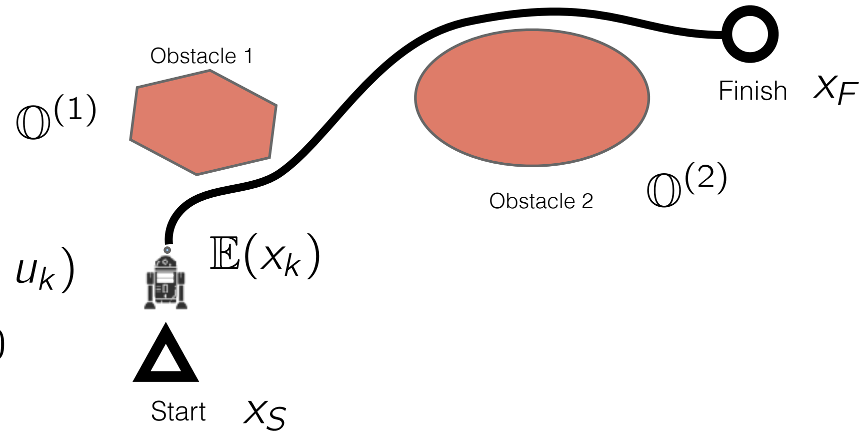

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& \|A^{(m)\top} \lambda_k^{(m)}\|_* \le 1, \; \lambda_k^{(m)} \succeq_{\mathcal{K}^*} 0, \\
& \text{for } k = 0, \dots, N, \; m = 1, \dots, M,
\end{aligned}$$

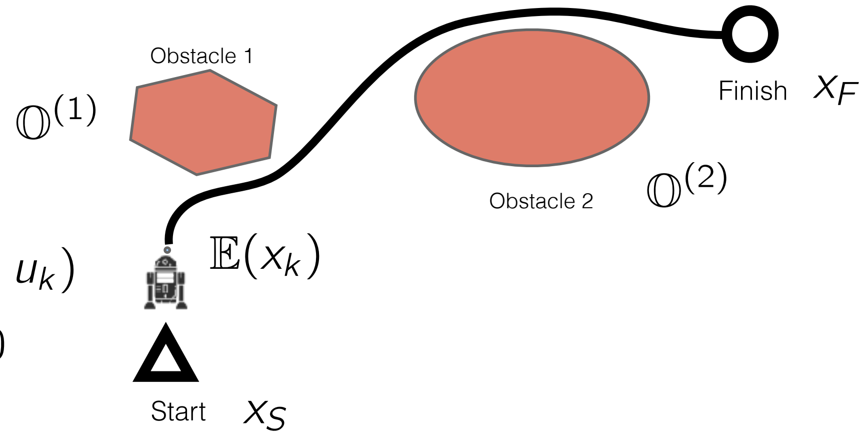
Dynamics

Collision constraints

If an appropriate cone $\mathcal{K}$ is used, the reformulation is smooth (but nonlinear)

Easy to add as a constraint in NLP solver

Distance Reformulation



$$x_{k+1} = f(x_k, u_k)$$
$$h(x_k, u_k) \leq 0$$

Collision free motion planning optimization problem

$$\min_{x,u,\boldsymbol{\lambda}} \sum_{k=0}^{N} \ell(x_k, u_k)$$

$$\text{s.t.} \quad x_0 = x_S, \; x_{N+1} = x_F,$$
$$x_{k+1} = f(x_k, u_k), \; h(x_k, u_k) \leq$$
$$(A^{(m)} p_k - b^{(m)})^\top \lambda_k^{(m)} > 0,$$
$$\|A^{(m)\top} \lambda_k^{(m)}\|_* \leq 1, \; \lambda_k^{(m)} \succeq_{\mathcal{K}^*} 0$$
$$\text{for } k = 0, \ldots, N, \; m = 1, \ldots, M,$$

Dynami... state

Collision constraints

If an appropriate cone $\mathcal{K}$ is used, the reformulation is smooth (but nonlinear)

Easy to add as a constraint in NLP solver

No information when overlapping

Signed distance reformulation

Theorem 2: Signed distance reformulation

If the obstacle is given by $\mathbb{O} = \{y \in \mathbb{R}^n \colon\ Ay \preceq_{\mathcal{K}} b\}$ and the ego shape is a point mass $\mathbb{E}(x) = p$, the requirement that the distance between the two sets is larger than a safety distance $\mathsf{d} \in \mathbb{R}$ is equivalent to the following constraints:

$$\mathrm{sd}(\mathbb{E}(x), \mathbb{O}) > \mathsf{d} \iff \exists \lambda \succeq_{\mathcal{K}^*} 0 \colon (A\,p - b)^\top \lambda > \mathsf{d},\ \|A^\top \lambda\|_* = 1$$

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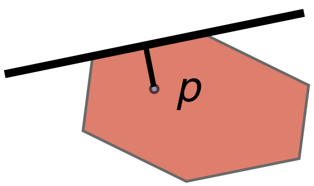
  - Reformulate the penetration as the minimum distance from any supporting hyperplane to $\mathbb{E}(x) = p$,



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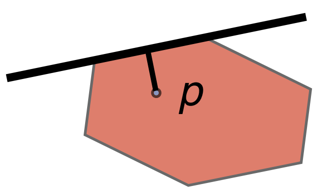

$$\mathrm{pen}(\mathbb{E}(x), \mathbb{O}) < \mathrm{p}_{\max} \iff \exists \lambda \succeq_{\mathcal{K}^*} 0 \colon (b - Ap)^\top \lambda < \mathrm{p}_{\max}, \ \|A^\top \lambda\|_* = 1$$

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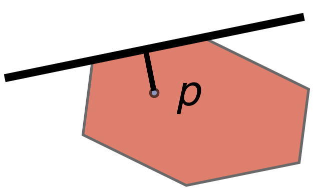

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  - Signed distance is $\mathrm{dist}(\mathbb{E}(x), \mathbb{O})$ if separated and $-\mathrm{pen}(\mathbb{E}(x), \mathbb{O})$ if overlapping

Signed distance Reformulation

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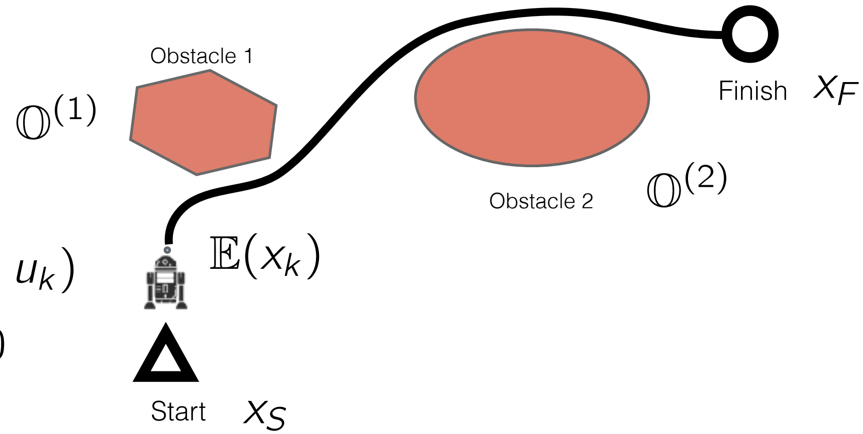

Minimum penetration motion planning optimization problem

$$
\begin{aligned}
\min_{x,u,s,\boldsymbol{\lambda}} \quad & \sum_{k=0}^{N} \left[\ell(x_k, u_k) + \kappa \cdot \sum_{m=1}^{M} s_k^{(m)} \right] && \text{Cost} \\
\text{s.t.} \quad & x_0 = x(0), \ x_{N+1} = x_F, && \text{Start and finish state} \\
& x_{k+1} = f(x_k, u_k), \ h(x_k, u_k) \leq 0, && \text{Dynamics} \\
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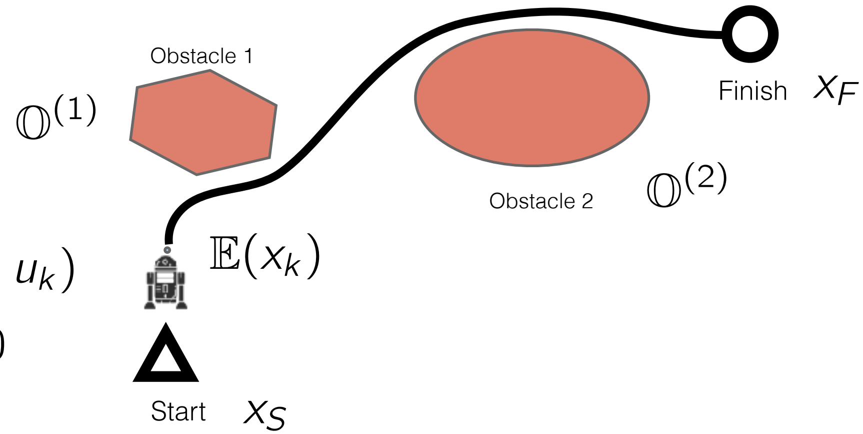
Signed distance Reformulation



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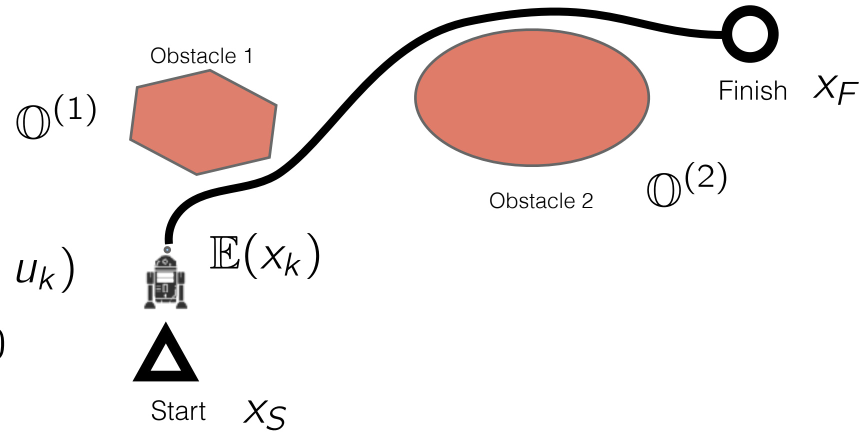

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Possible to use soft constraints
Solver has information about penetration

Start and finish state
Dynamics

Collision constraints

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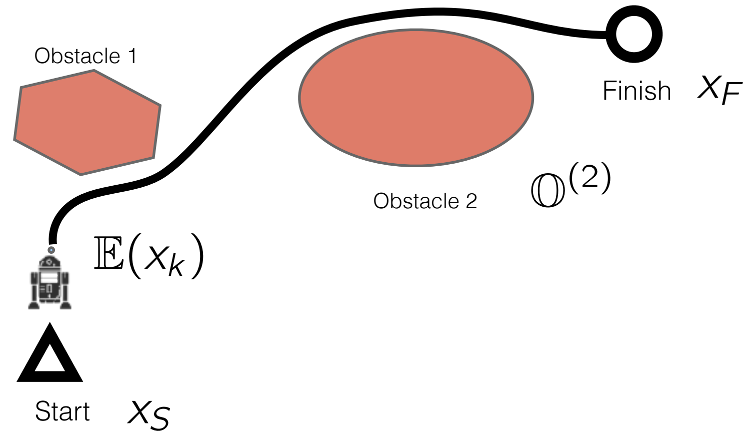

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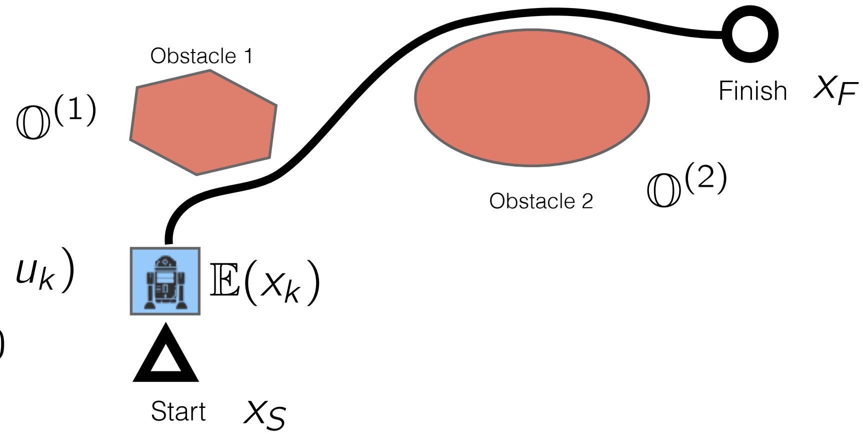
Possible to use soft constraints

Solver has information about penetration

Additional non-convex constraint

Collision constraints

Full-sized collision avoidance



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Smooth collision constraint reformulation

- Full-sized ego shape: ego shape is a rotated and translated convex set

Full-sized collision avoidance

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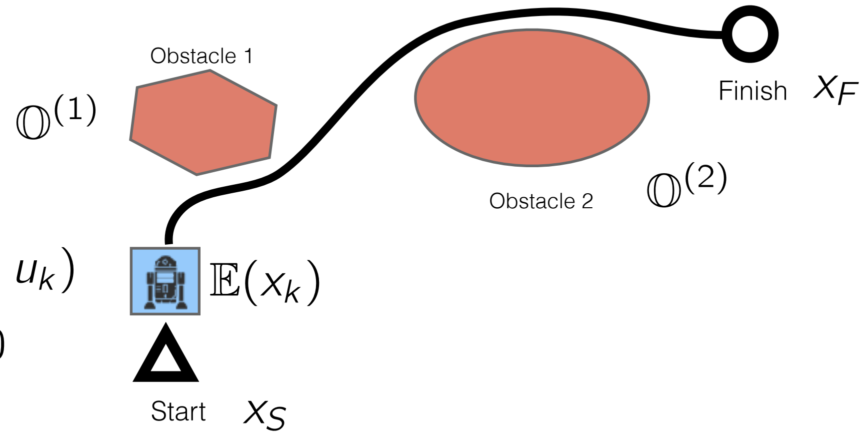

Smooth collision constraint reformulation

- Full-sized ego shape: ego shape is a rotated and translated convex set

$$\mathbb{E}(x_k) = R(x_k)\mathbb{B} + t(x_k), \quad \mathbb{B} := \{y : Gy \preceq_{\bar{\mathcal{K}}} g\}$$

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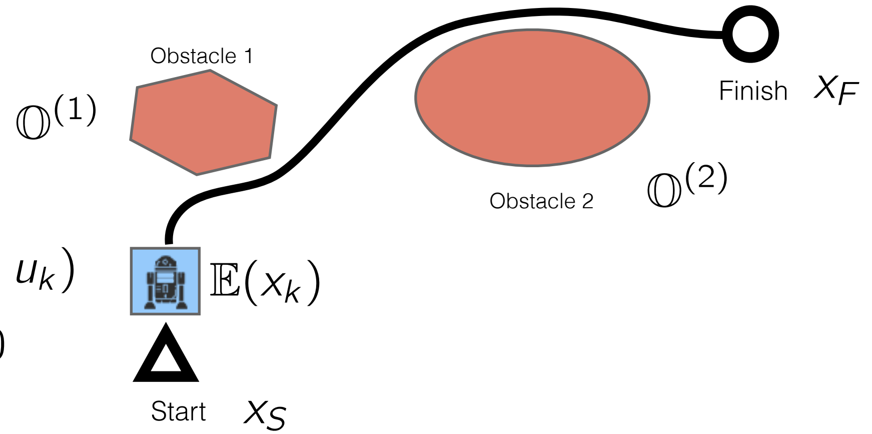

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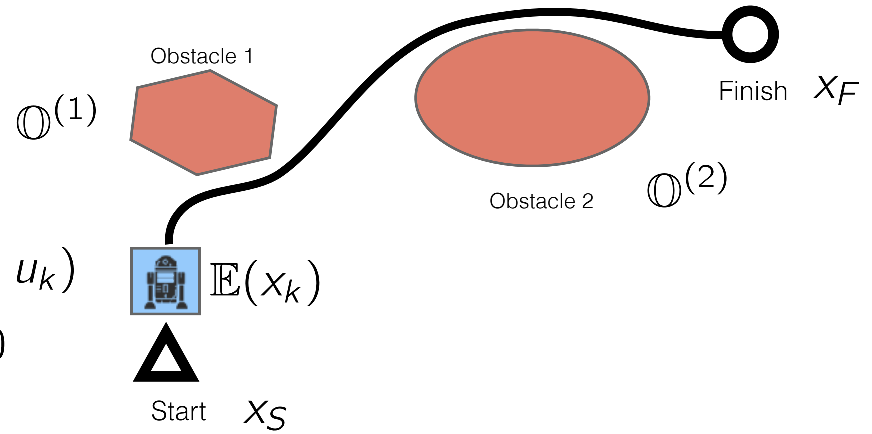
  - Ego shape is a circle and obstacle is a circle or ellipse

Full-sized collision avoidance



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- When is it "easy" to handle the collision constraint $\mathbb{E}(x) \cap \mathbb{O} = \emptyset$?
  - Ego shape is a circle and obstacle is a circle or ellipse
- We show that the collision constraint can be reformulated as a smooth but non-convex constraint by reformulating the distance and signed distance

Distance reformulation

Theorem 3: Full-sized distance reformulation

If the obstacle is given by $\mathbb{O} = \{y \in \mathbb{R}^n \colon Ay \preceq_{\mathcal{K}} b\}$ and the ego shape is a point mass $\mathbb{E}(x_k) = R(x)\mathbb{B} + t(x)$, the requirement that the distance between the two sets is larger than a distance $\mathsf{d}_{\min} \geq 0$ is equivalent to the following constraints:

$$\mathrm{dist}(\mathbb{E}(x), \mathbb{O}) > \mathsf{d}_{\min} \iff \exists \lambda \succeq_{\mathcal{K}^*} 0, \mu \succeq_{\bar{\mathcal{K}}^*} 0:$$

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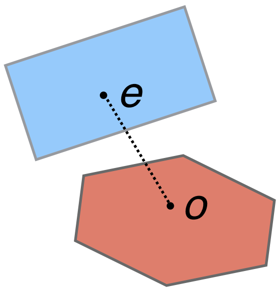
- Proof sketch:
  - $\mathrm{dist}(\mathbb{E}(x), \mathbb{O})$ is given by the following convex program:

$$\mathrm{dist}(\mathbb{E}(x), \mathbb{O}) = \min_{e,o} \{\|e - o\| \colon\ Ao \preceq_{\mathcal{K}} b, e \in \mathbb{E}(x)\}$$



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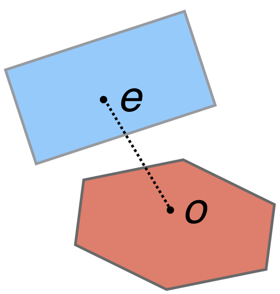

  - By strong duality the dual is also equal to the distance

$$\mathrm{dist}(\mathbb{E}(x), \mathbb{O}) = \max_{\lambda,\mu} \{-g^\top \mu + (At(x) - b)^\top \lambda \colon G^\top \mu + R(x)^\top A^\top \lambda = 0,$$
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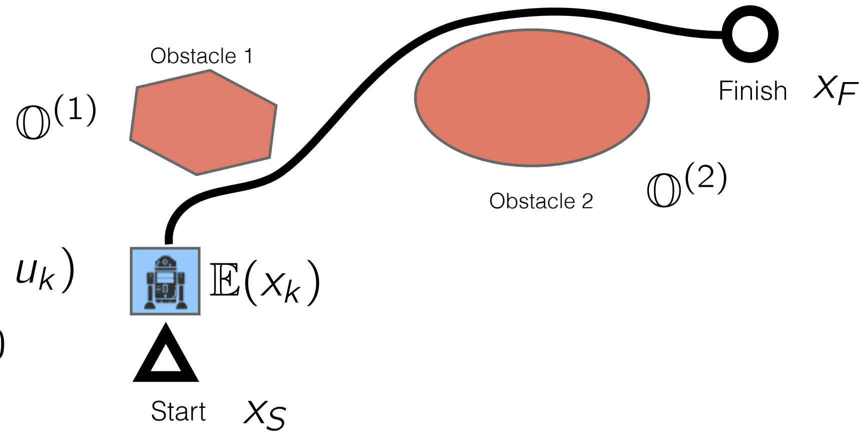

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\text{s.t.} \quad & x_0 = x(0),\ x_{N+1} = x_F, && \text{Start and finish state} \\
& x_{k+1} = f(x_k, u_k),\ h(x_k, u_k) \leq 0, && \text{Dynamics} \\
& -g^\top \mu_k^{(m)} + (A^{(m)} t(x_k) - b^{(m)})^\top \lambda_k^{(m)} > 0, && \\
& G^\top \mu_k^{(m)} + R(x_k)^\top A^{(m)\top} \lambda_k^{(m)} = 0, && \text{Collision constraints} \\
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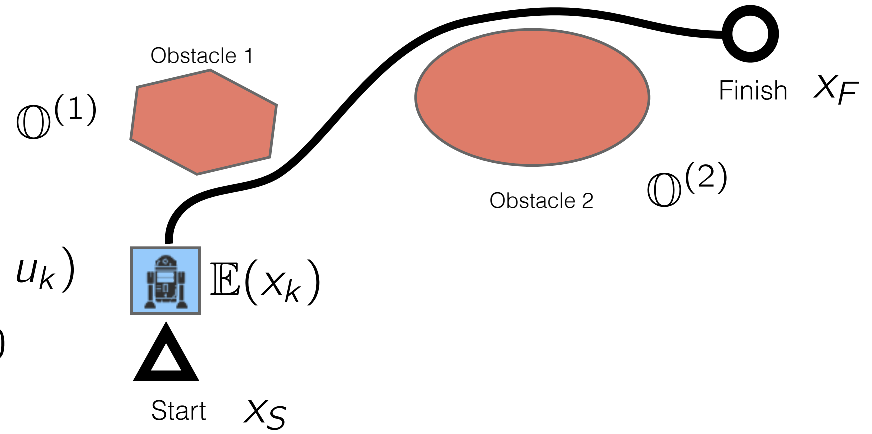

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Signed Distance reformulation

Theorem 4: Full-sized signed distance reformulation

If the obstacle is given by $\mathbb{O} = \{y \in \mathbb{R}^n : Ay \preceq_{\mathcal{K}} b\}$ and the ego shape is a point mass $\mathbb{E}(x_k) = R(x)\mathbb{B} + t(x)$, the requirement that the distance between the two sets is larger than a distance $\mathsf{d} \in \mathbb{R}$ is equivalent to the following constraints:

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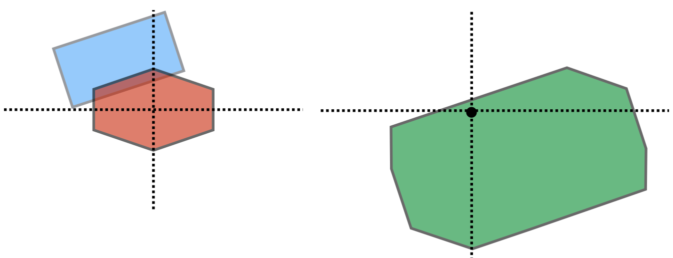
    $$\mathrm{pen}(\mathbb{E}(x), \mathbb{O}) = \mathrm{pen}(0, \mathbb{O} - \mathbb{E}(x))$$



  - Where $\mathbb{O} - \mathbb{E}(x) := \{o - e \colon\ o \in \mathbb{O},\ e \in \mathbb{E}(x)\}$ is the Minkowski difference
    - Minkowski difference of two convex sets is convex
    - Minkowski difference only contains **0** if sets overlap

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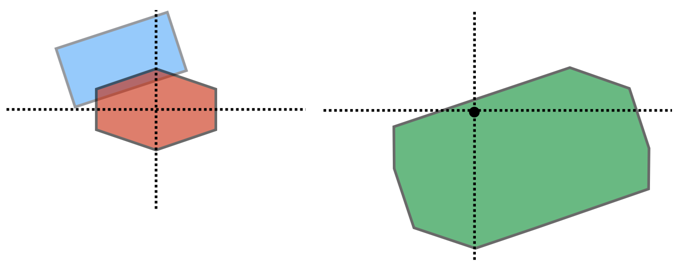
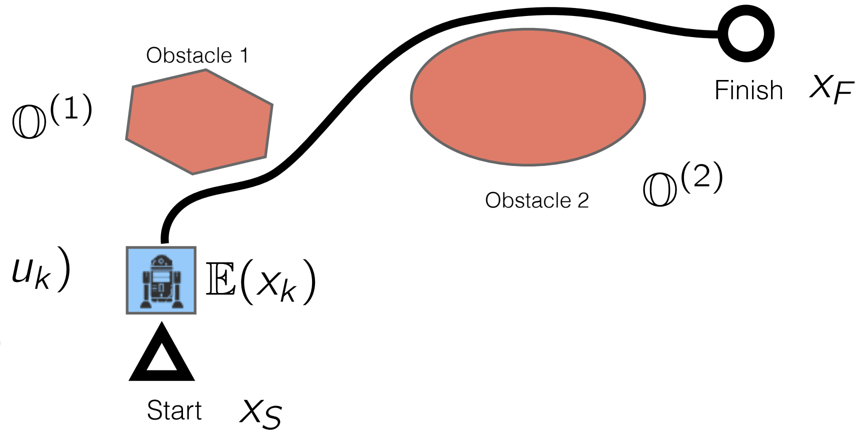
    - Minkowski difference of two convex sets is convex
    - Minkowski difference only contains **0** if sets overlap
  - Back to point mass penetration case



Distance Reformulation

$$x_{k+1} = f(x_k, u_k)$$

$$h(x_k, u_k) \leq 0$$



Minimum penetration motion planning optimization problem

$$\begin{aligned}
\min_{\mathrm{x,u,s},\boldsymbol{\lambda},\boldsymbol{\mu}} \quad & \sum_{k=0}^{N} \left[\ell(x_k, u_k) + \kappa \cdot \sum_{m=1}^{M} s_k^{(m)} \right] && \text{Cost} \\
\text{s.t.} \quad & x_0 = x_S, \; x_{N+1} = x_F, && \text{Start and finish state} \\
& x_{k+1} = f(x_k, u_k), \; h(x_k, u_k) \leq 0, && \text{Dynamics} \\
& -g^\top \mu_k^{(m)} + (A^{(m)} t(x_k) - b^{(m)})^\top \lambda_k^{(m)} > -s_k^{(m)}, \\
& G^\top \mu_k^{(m)} + R(x_k)^\top {A^{(m)}}^\top \lambda_k^{(m)} = 0, && \text{Collision constraints} \\
& \| {A^{(m)}}^\top \lambda_k^{(m)} \|_* = 1, \\
& s_k^{(m)} \geq 0, \; \lambda_k^{(m)} \succeq_{\mathcal{K}^*} 0, \; \mu_k^{(m)} \succeq_{\bar{\mathcal{K}}^*} 0, \\
& \text{for } k = 0, \ldots, N, \; m = 1, \ldots, M.
\end{aligned}$$

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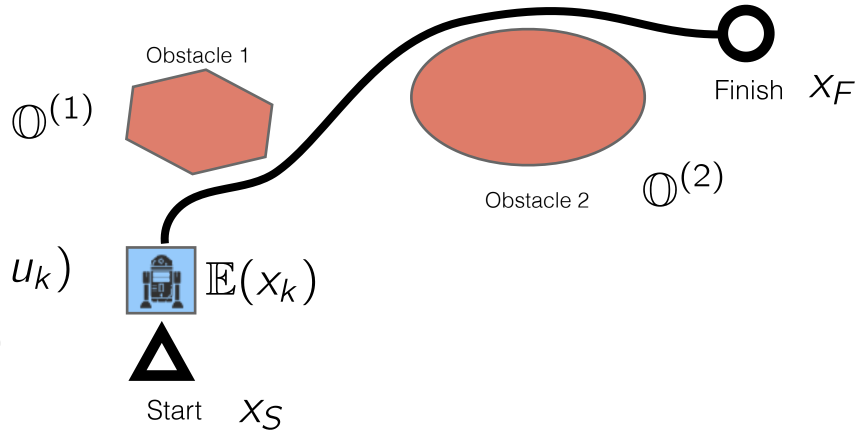

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\end{aligned}
$$

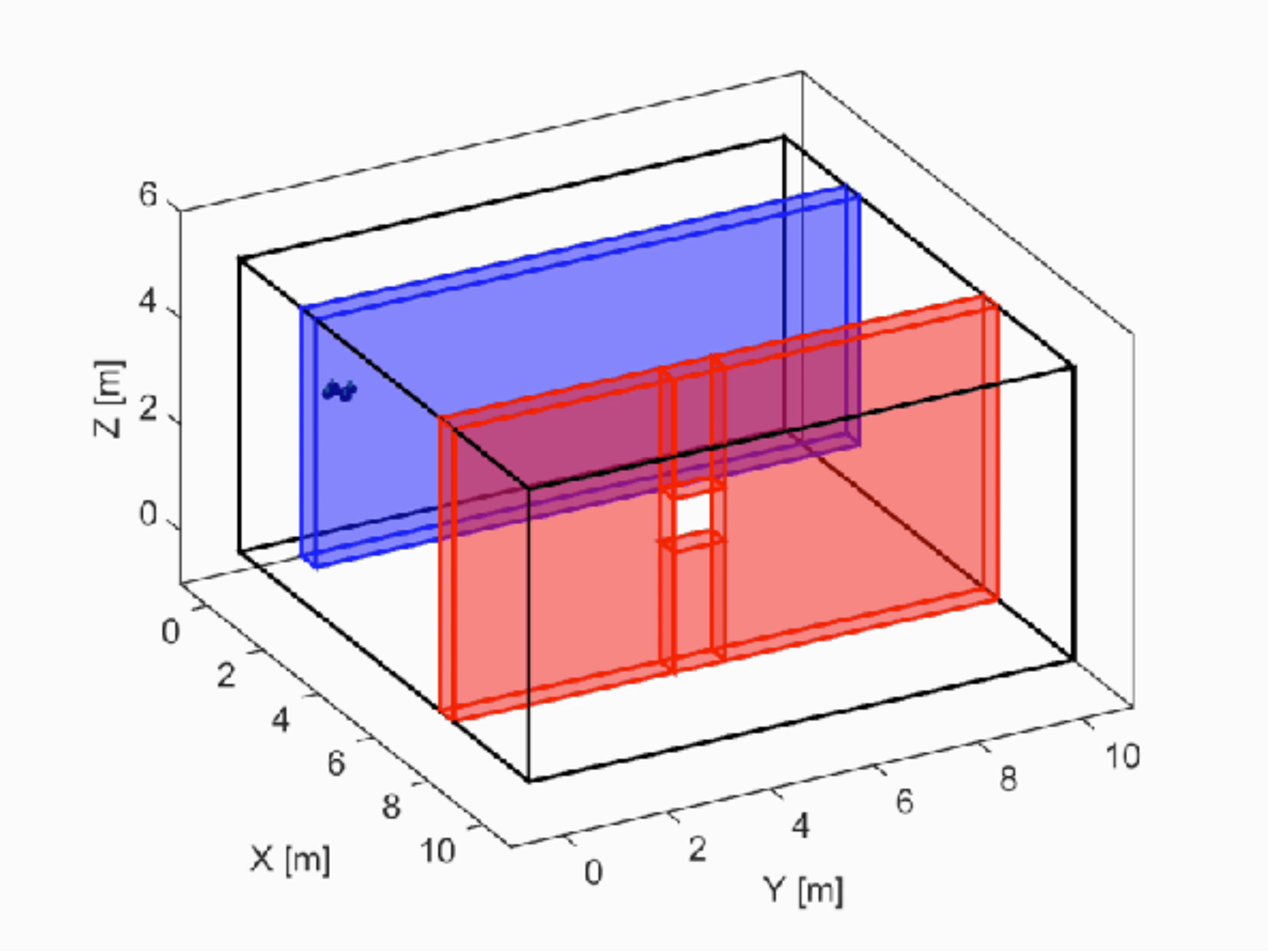
Cost

Start and finish state

Dynamics

Collision constraints

Quadcopert motion planning



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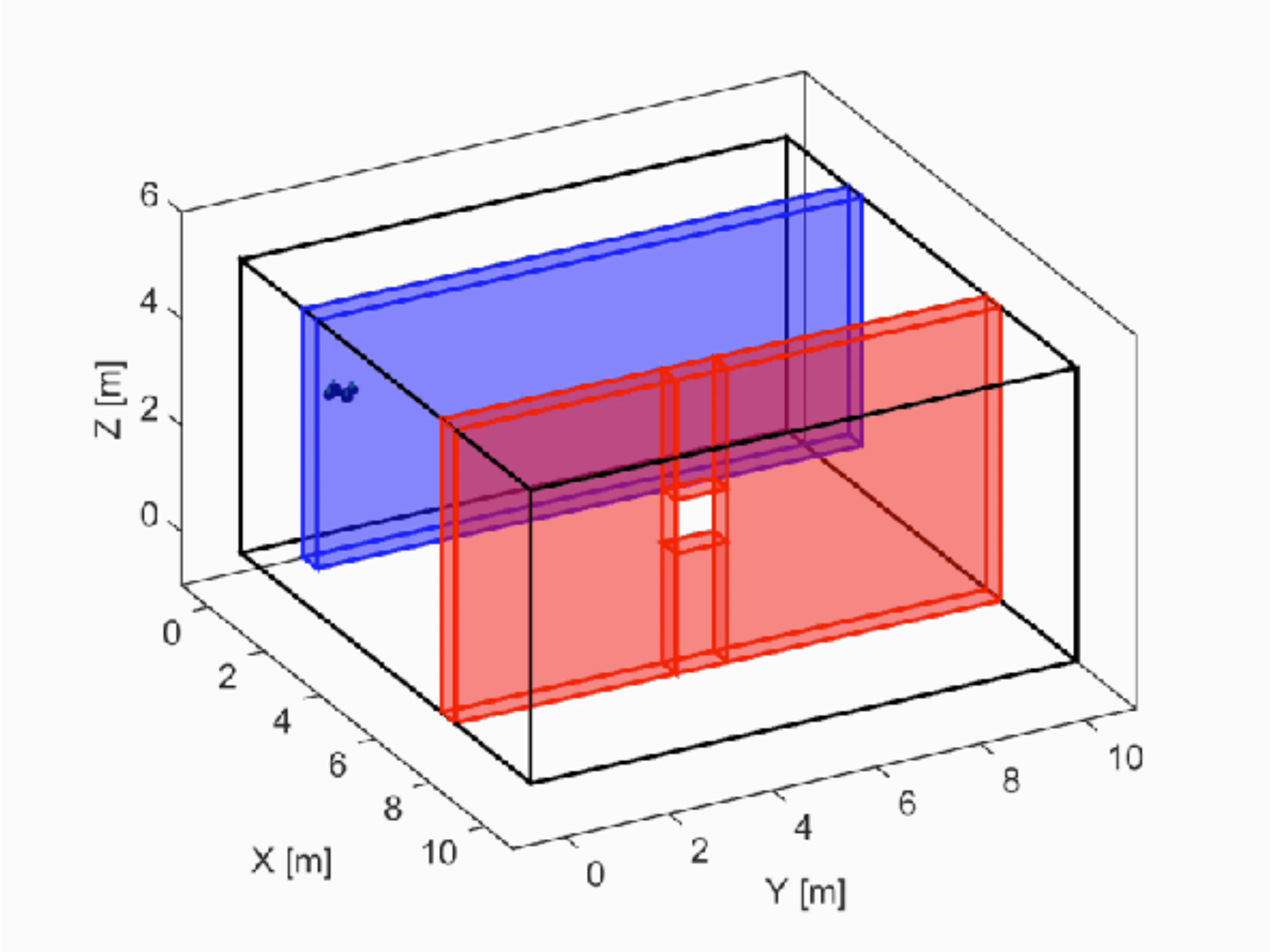

Problem definition

- Model:
  - Full 12-state quadcopter model - rotor speeds as inputs [Meilinger]
- Input-state constraints:
  - Bounds on states and inputs
- Cost:
  - tradeoff between minimum time and minimum input

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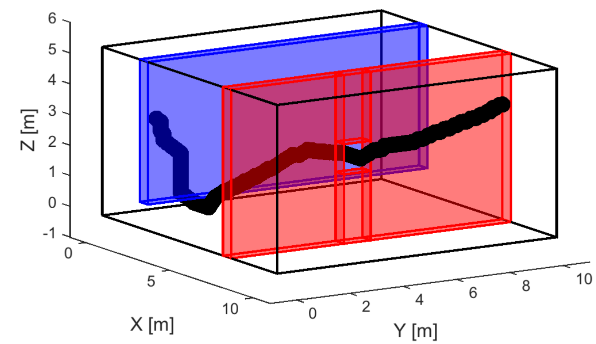
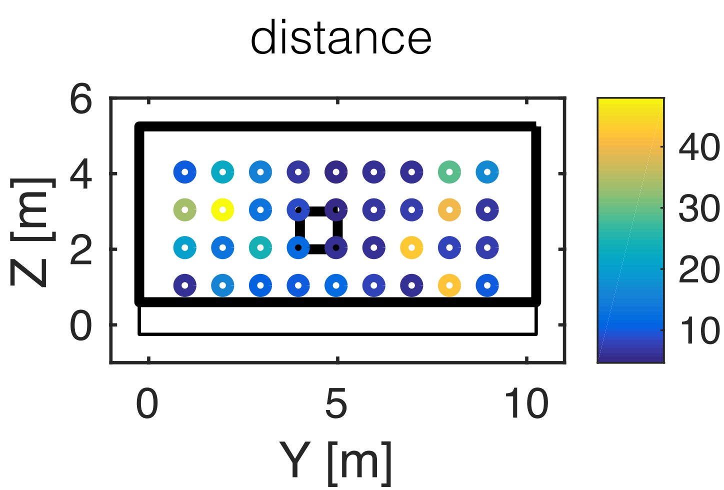
- Obstacle avoidance:
  - Point mass ego shape with a safety distance to consider size of the quadcopter
  - Obstacles are five 3D boxes

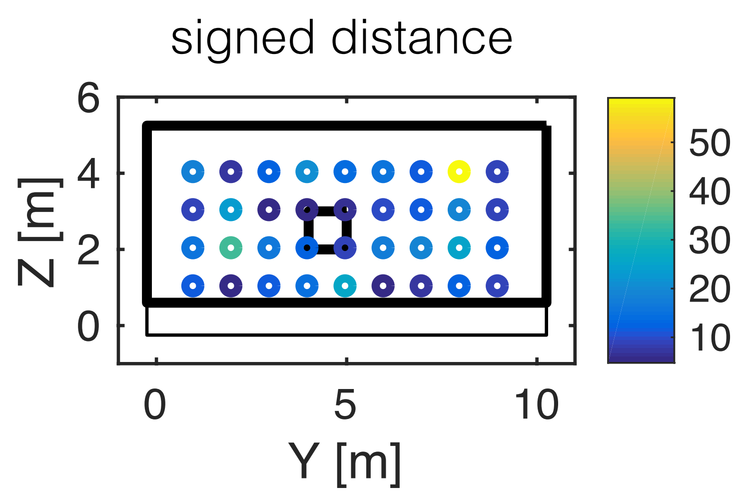
Results

- Warm start using shortest path problem
  - A* is used to solve the 3-D shortest path problem
  - A* also determines horizon length *N*
  - Zero velocities and angles warm start



- *IPOPT* as NLP solver and *Julia/JuMP* as interface
- Solved for 36 different final positions
  - *N* between 100 - 129, *Ts* limited between 0.125 and 0.375 s





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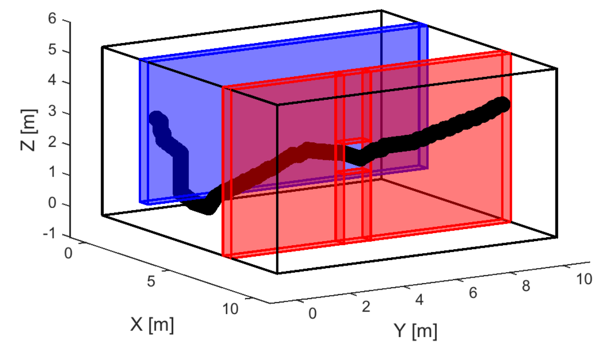

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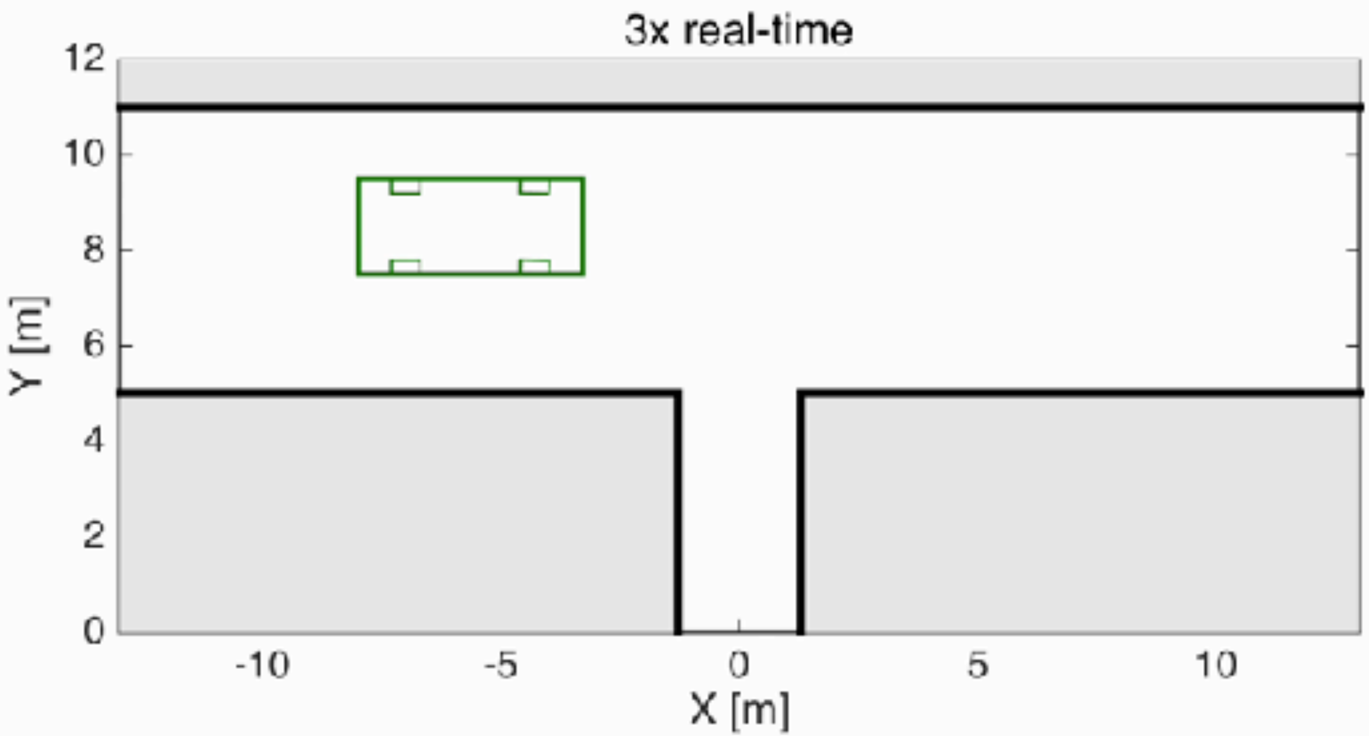
Quadcopter navigation	min	max	mean
warm start (A*)	0.5724 s	2.8157 s	1.6207 s
distance formulation	4.6806 s	47.9762 s	14.9716 s
signed distance formulation	4.7638 s	59.1031 s	14.3962 s

Autonomous parking



- Considering ego-shape is necessary
  - Approximate ego-shape as a ball leads to an infeasible problem

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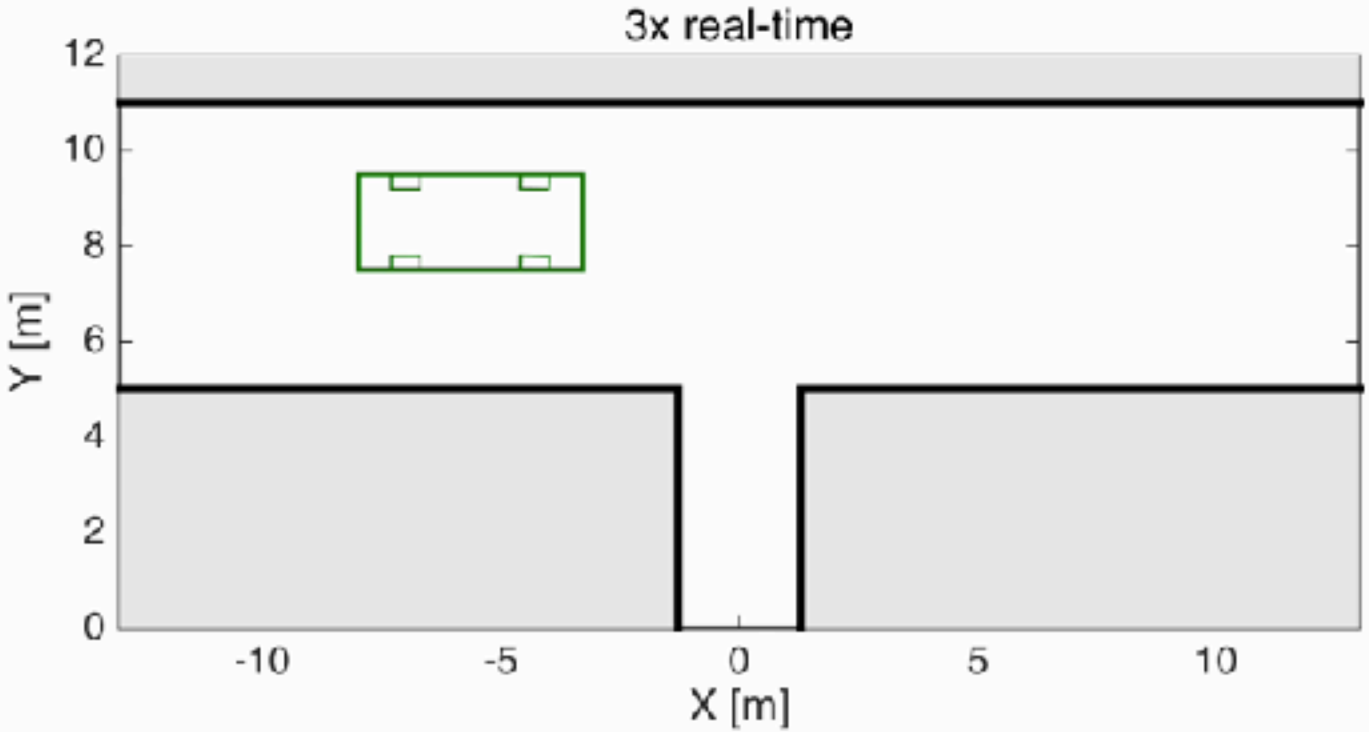

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Problem definition

- **Model:**
  - 4-state kinematic car model
  - steering and acceleration input

$$\dot{X} = v\cos(\varphi)$$
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$$\dot{\varphi} = \frac{v\tan(\delta)}{L}$$
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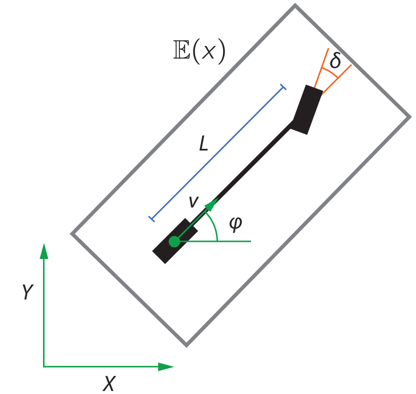

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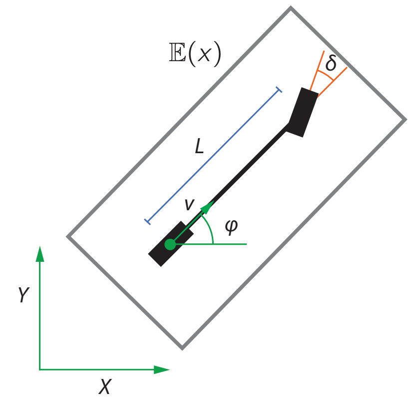

- **Input-state constraints:**
  - Bounds on steering $\boldsymbol{\delta}$, steering rate $\boldsymbol{\Delta\delta}$, and acceleration $\boldsymbol{a}$

- **Cost:**
  - tradeoff between minimum time, minimum input, and minimum input rate

$$J = qNT_{\text{opt}} + \sum_{k=0}^{N-1} u_k^T R u_k + \Delta u_k^T R_\Delta \Delta u_k$$

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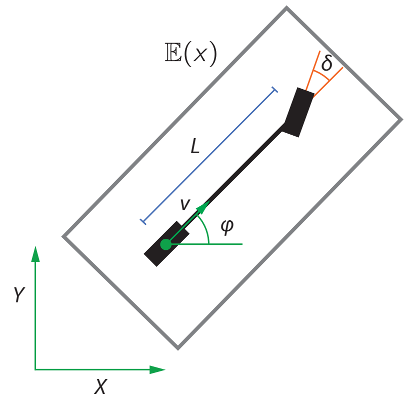

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- Obstacle avoidance:
  - Box shape for car
  - 5 half-spaces for reverse and 6 for parallel parking

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  - Warm-start should avoid obstacles, and
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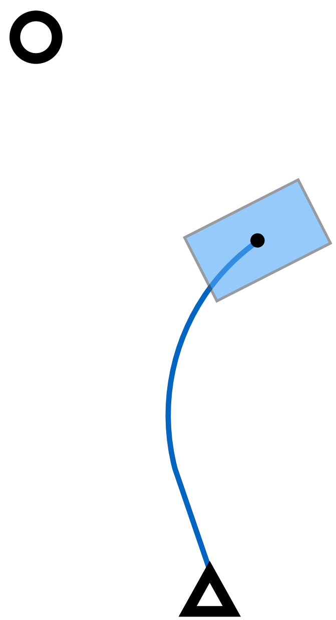

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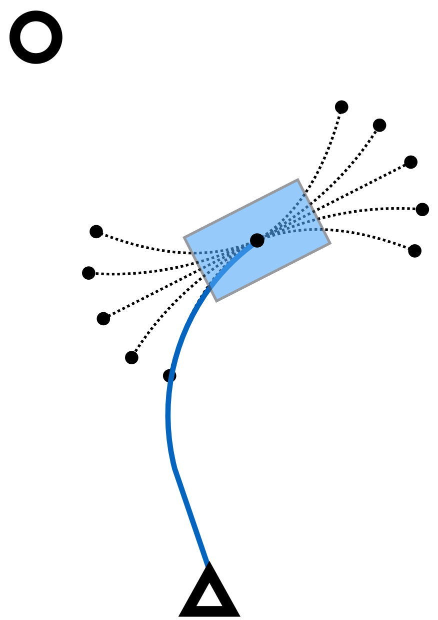

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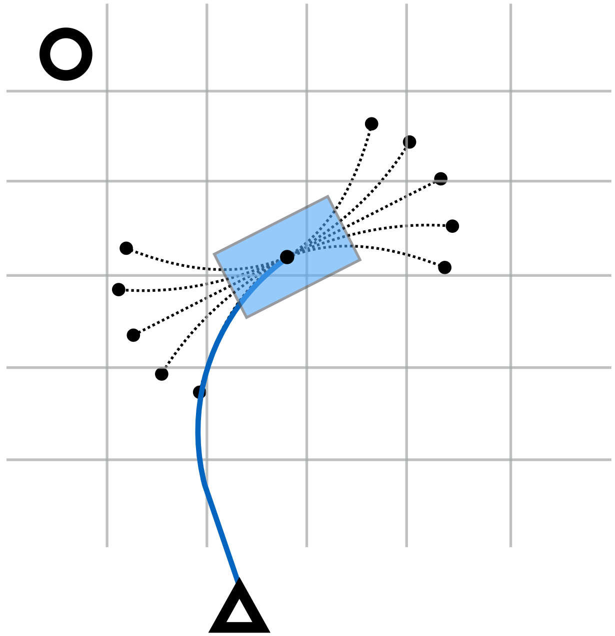

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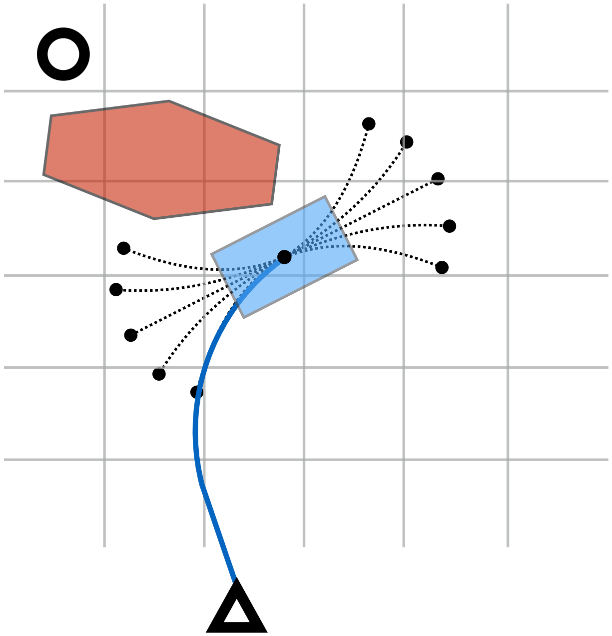

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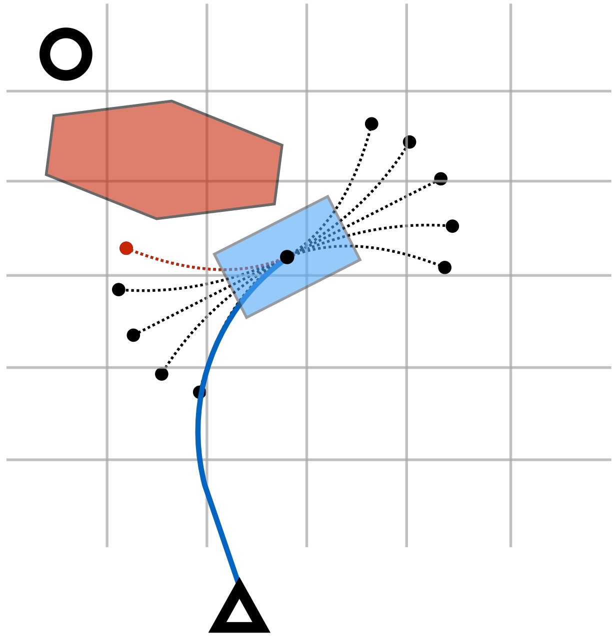

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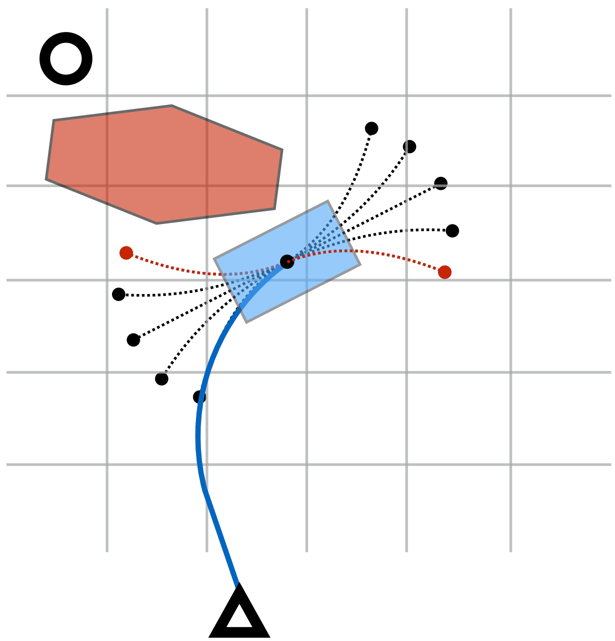

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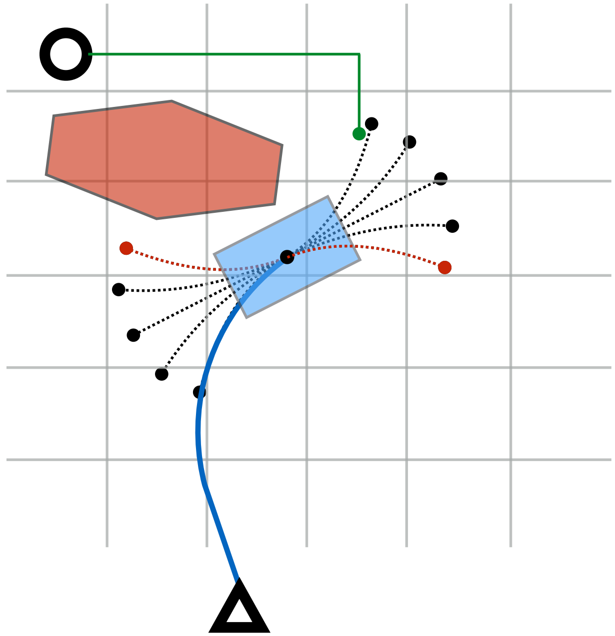

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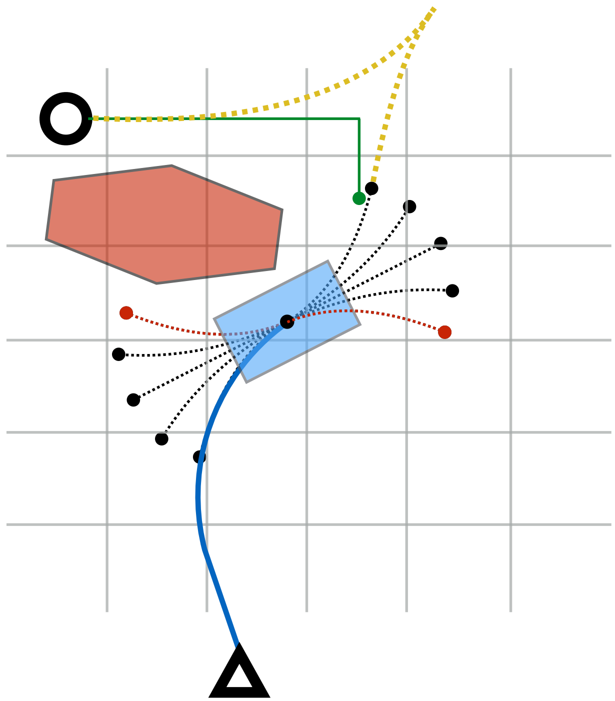

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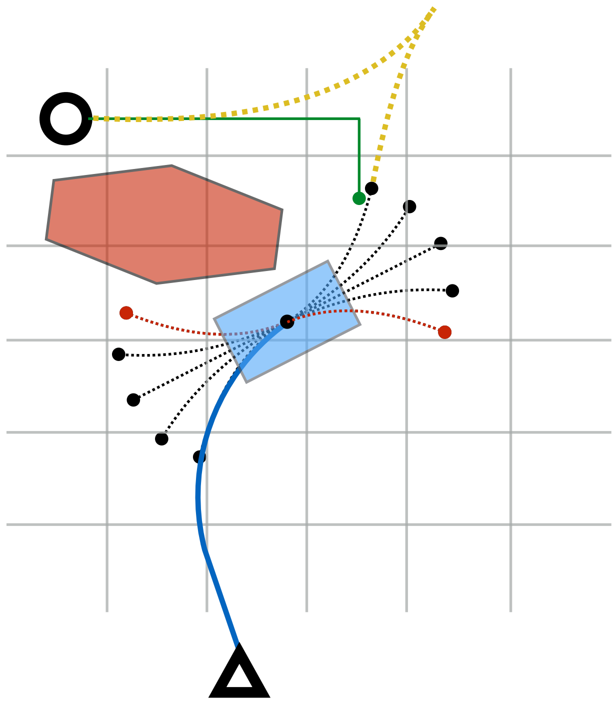

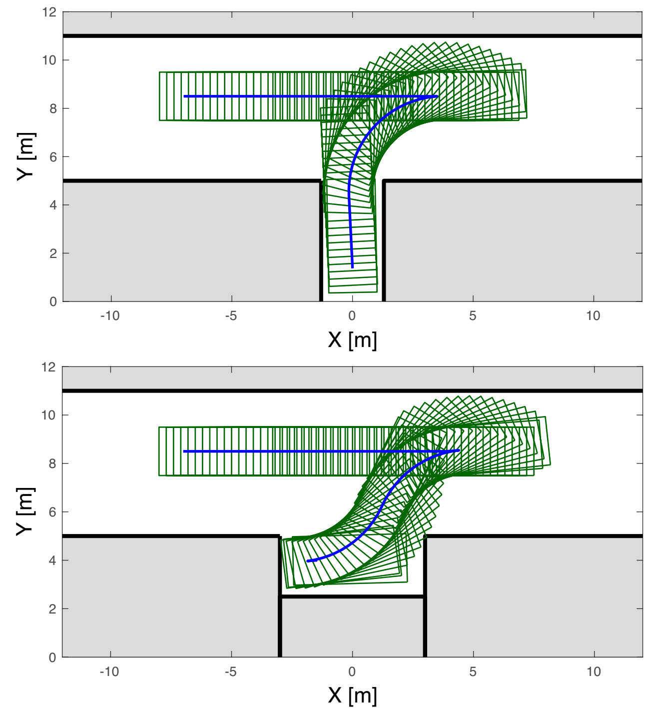

Results

- Warm start using Hybrid-A*
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- *IPOPT* as NLP solver and *Julia/JuMP* as interface
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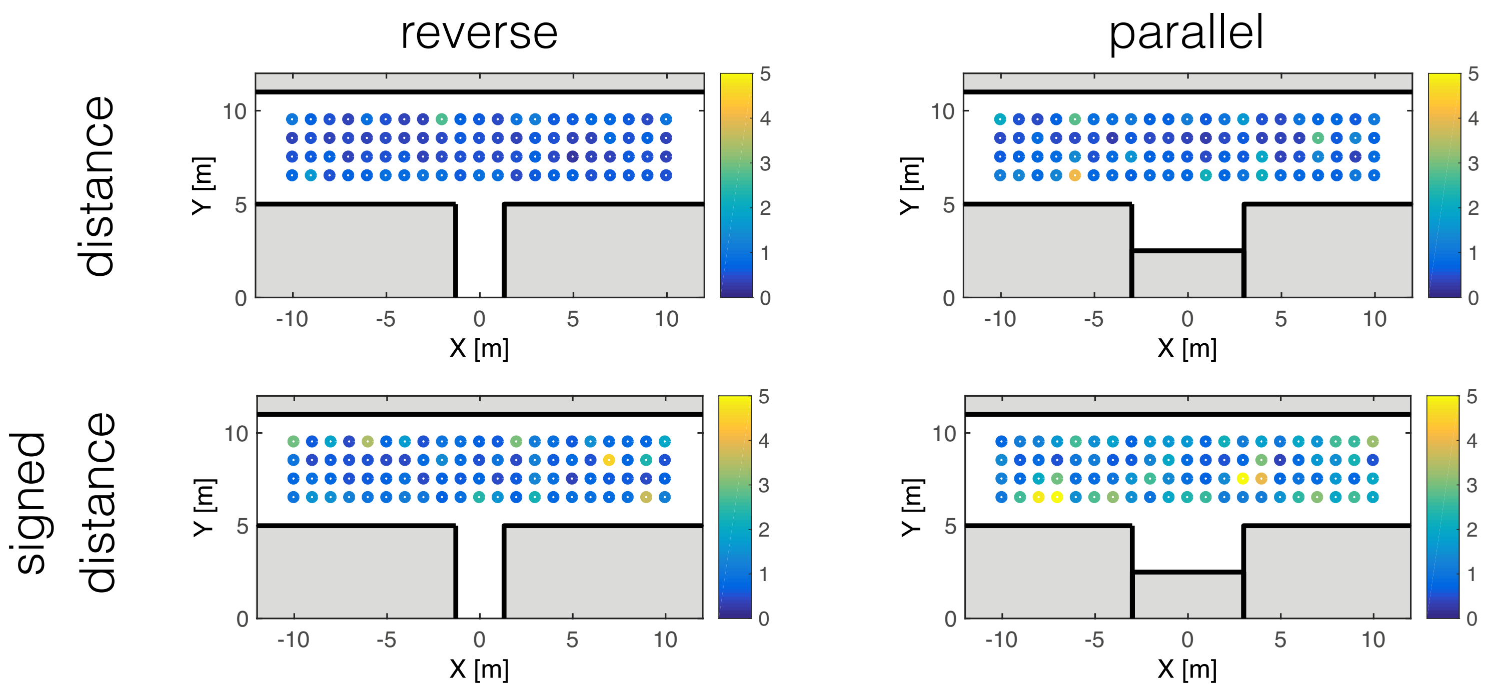

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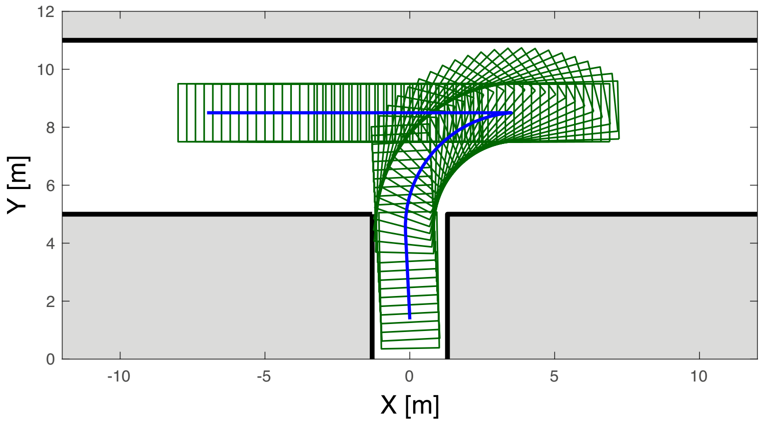
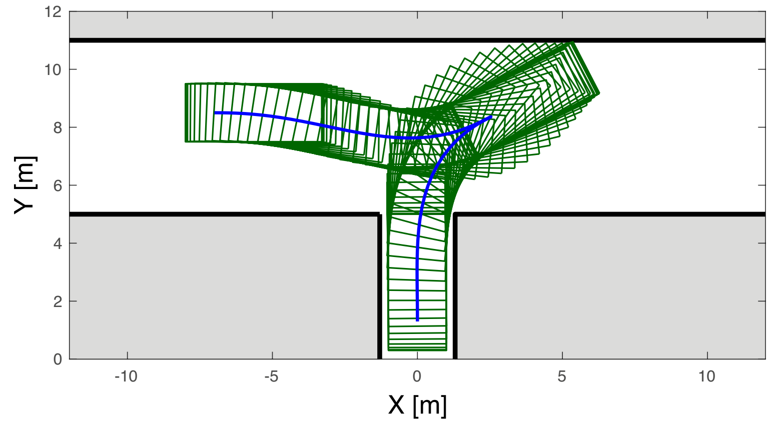
	min	max	mean
Reverse Parking			
warm start (Hybrid A*)	0.0315 s	3.2230 s	0.5491 s
distance formulation	0.2111 s	2.7166 s	0.6046 s
signed distance formulation	0.3200 s	4.4840 s	1.0344 s
Parallel Parking			
warm start (Hybrid A*)	0.0421 s	2.4766 s	0.3012 s
distance formulation	0.2561 s	3.9885 s	0.8682 s
signed distance reformulation	0.3850 s	6.7266 s	1.6703 s

Driveability

- Is it worth to use this approach or is a Hybrid A* good enough





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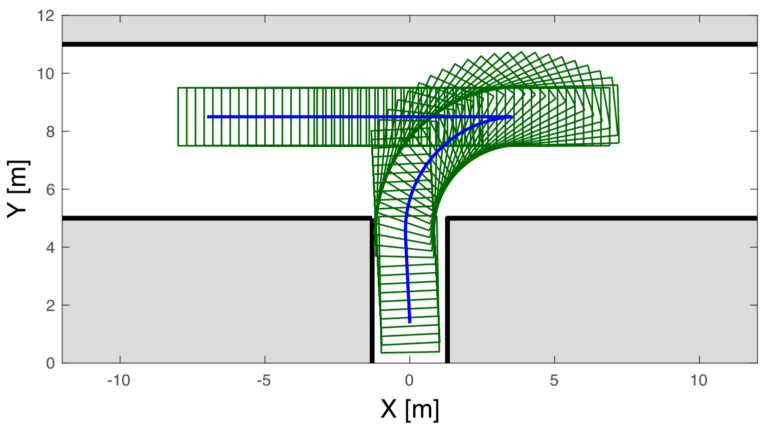

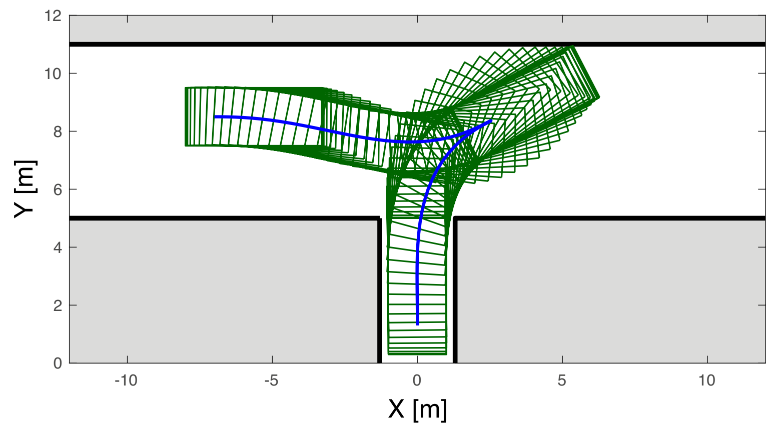

- How well can a path following controller follow the trajectory
  - Velocity P-Controller based on position along the path
  - Lateral path-following LQR

$$\dot{e} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} e + \begin{bmatrix} 0 \\ \frac{v}{L} \end{bmatrix} \delta, \quad \delta = -Ke + \delta_{\mathrm{ff}}(s)$$

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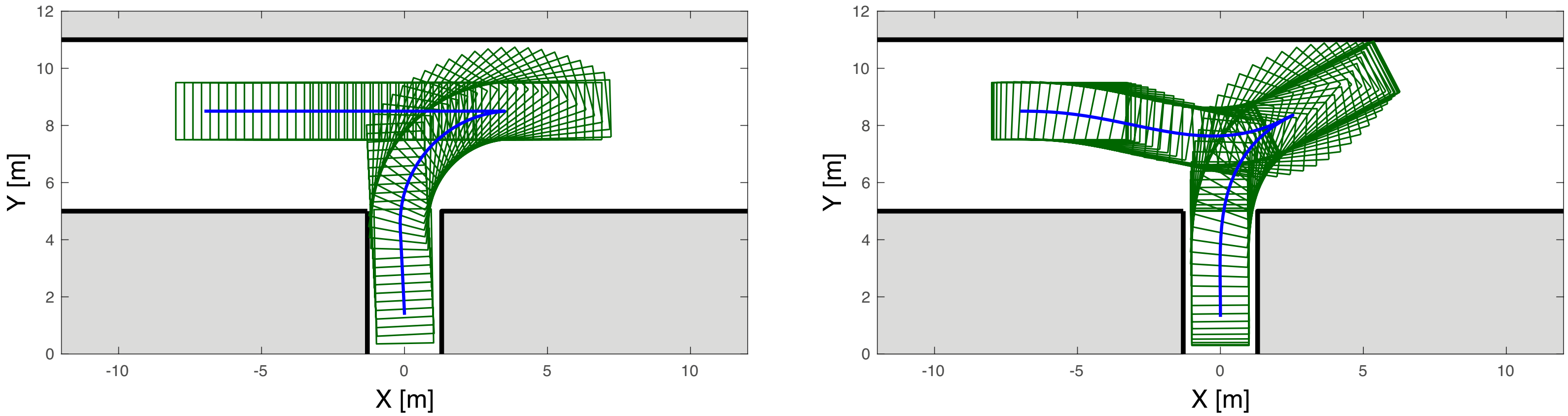

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- Hybrid A* neglects longitudinal dynamics and rate constraints
  - Accurate path following is only possible with a slow velocity profile

Driveability

- Reduce maneuvre time by over 50%, while improving the accuracy
  - OBCA trajectory considers full model and actuator limits,
  - resulting in smooth and easy to follow trajectories
  - gives accurate feedforward terms (no heuristic needed as for hybrid A*),

	min	max	mean
Reverse Parking			
Maneuver time Hybrid A*	36.9 s	75.9 s	55.2 s
Maneuver time OBCA	14.1 s	34.9 s	24.2 s
Max tracking error Hybrid A*	0.005 m	0.120 m	0.069 m
max tracking error OBCA	0.038 m	0.088 m	0.058 m
Parallel Parking			
Maneuver time Hybrid A*	51.1 s	131.9 s	86.5 s
Maneuver time OBCA	17.5 s	67.4 s	39.6 s
Max tracking error Hybrid A*	0.037 m	0.145 m	0.086 m
Max deviation OBCA	0.050 m	0.133 m	0.074 m

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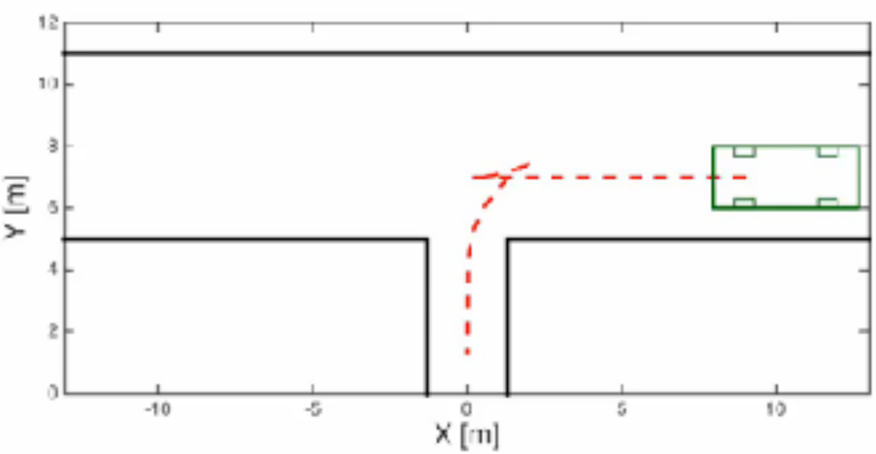

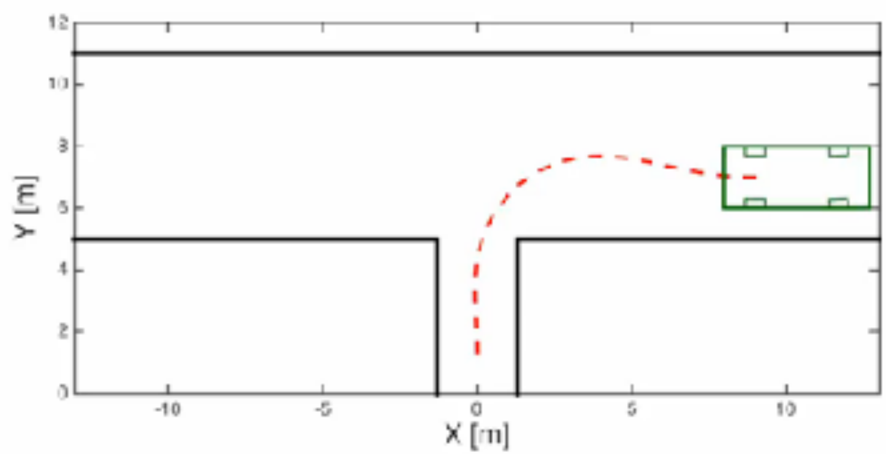

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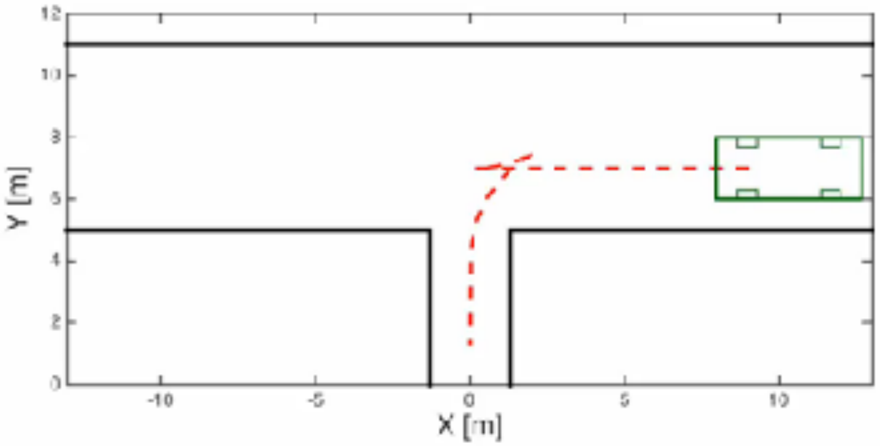
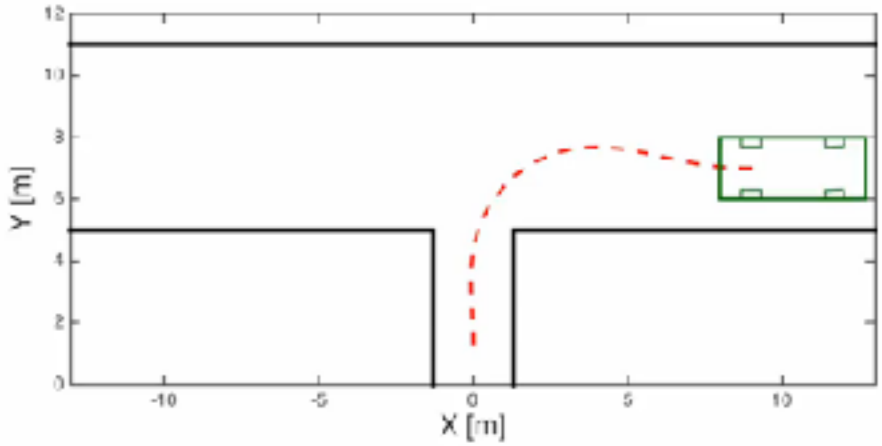
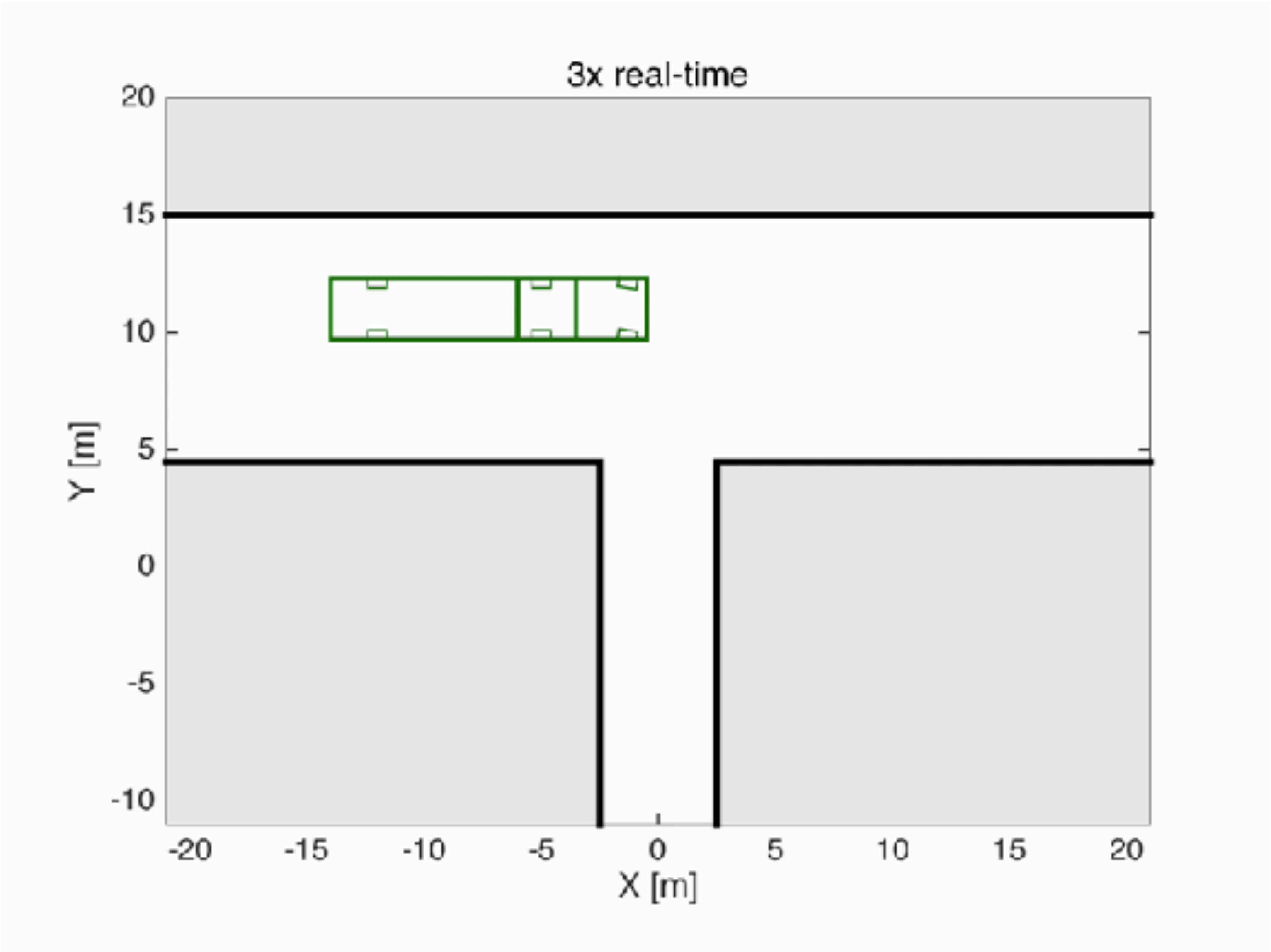




Conclusion

- Novel method for optimization-based collision avoidance
- Results in smooth and easy to implement constraints
- Showed the efficiency of the approach on a quadcopter and autonomous parking example
- On of the big challenges is finding a good warm-start

Truck parking - Questions



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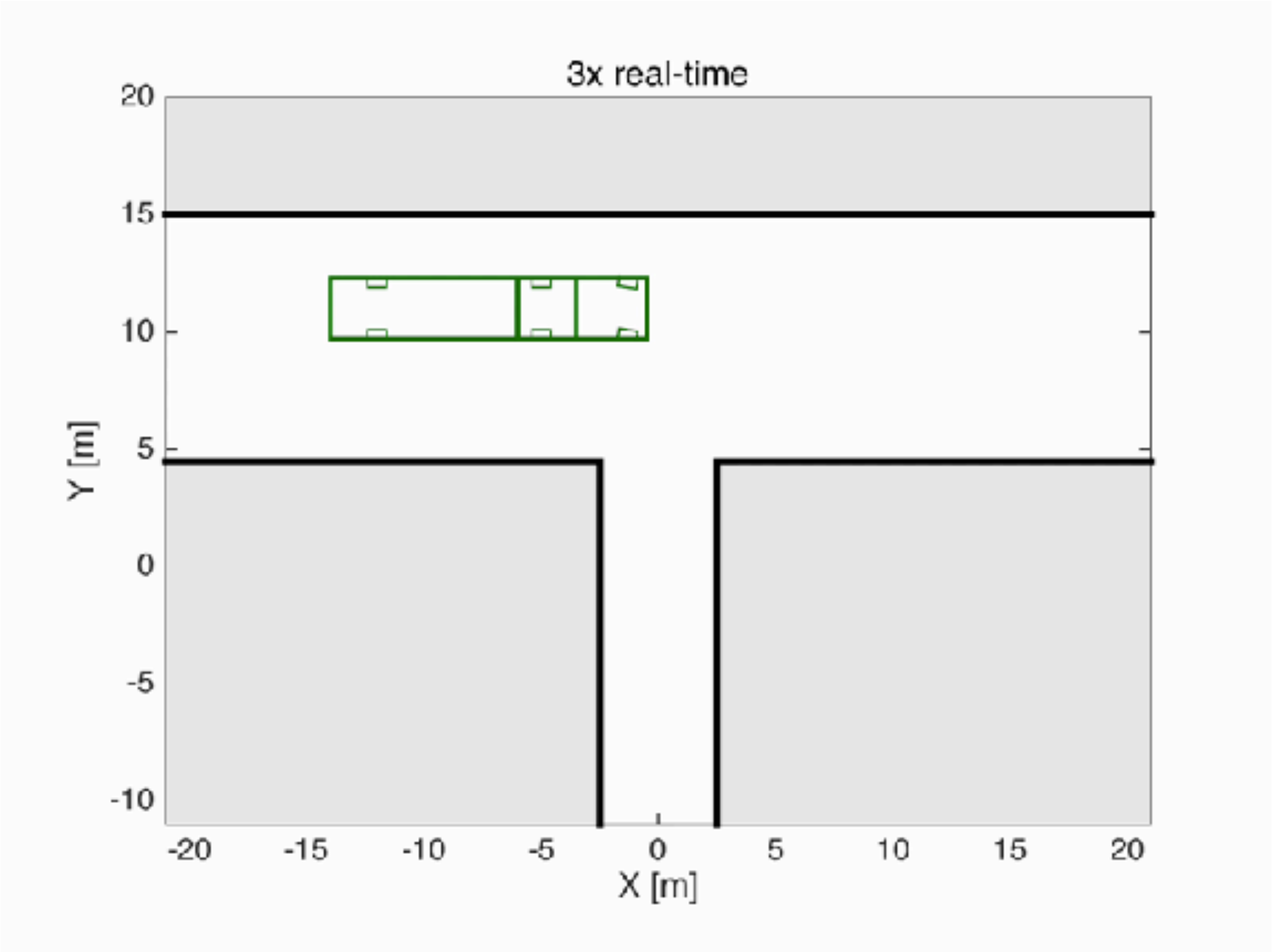
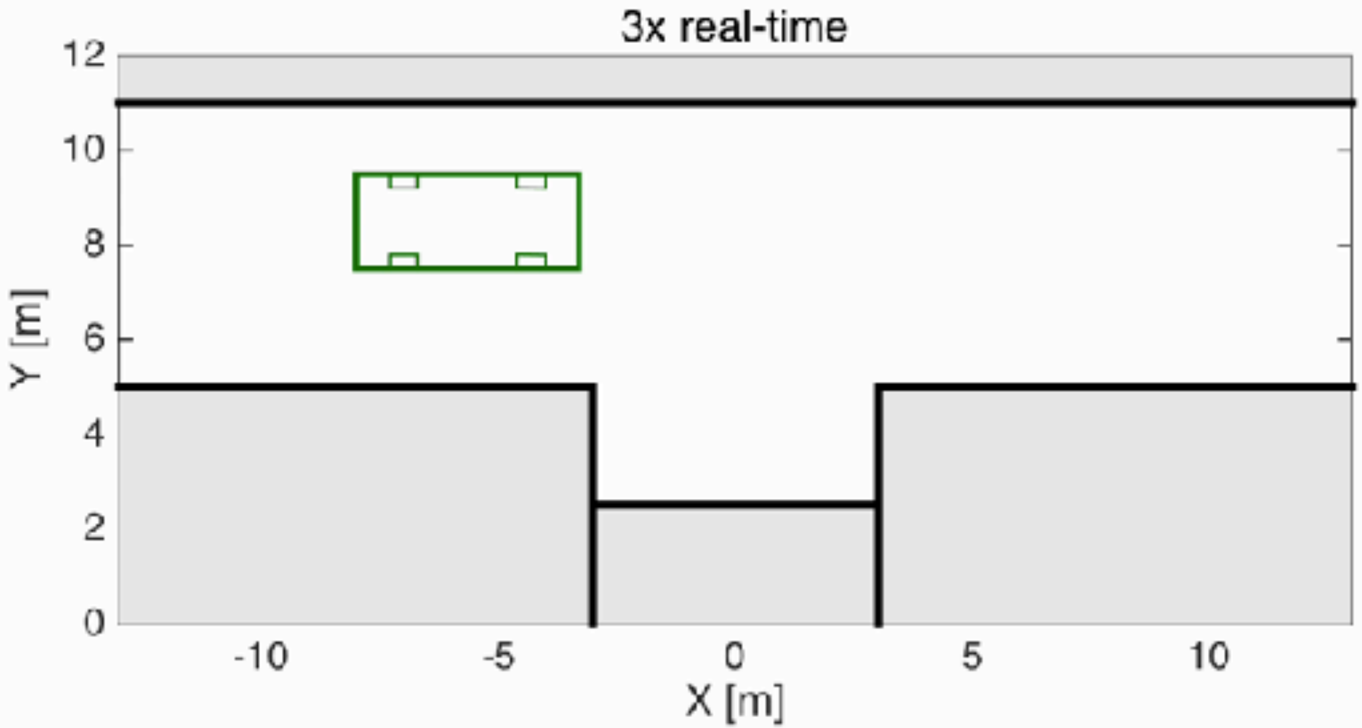


Parallel Parking



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